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Jumps of computably enumerable equivalence relations ☆

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ABSTRACT

We study computably enumerable equivalence relations (or, ceers), under computable reducibility $\leq$, and the halting jump operation on ceers. We show that every jump is uniform join-irreducible, and thus join-irreducible. Therefore, the uniform join of two incomparable ceers is not equivalent to any jump. On the other hand there exist ceers that are not equivalent to jumps, but are uniform join-irreducible: in fact above any non-universal ceer there is a ceer which is not equivalent to a jump, and is uniform join-irreducible. We also study transfinite iterations of the jump operation. If a is an ordinal notation, and E is a ceer, then let $E^{(a)}$ denote the ceer obtained by transfinitely iterating the jump on E along the path of ordinal notations up to a. In contrast with what happens for the Turing jump and Turing reducibility, where if a set X is an upper bound for the A-arithmetical sets then $X^{(2)}$ computes $A^{(\omega)}$, we show that there is a ceer R such that $R \geq \mathrm{Id}^{(n)}$, for every finite ordinal n, but, for all k, $R^{(k)} \not\geq \mathrm{Id}^{(\omega)}$ (here Id is the identity equivalence relation). We show that if a, b are notations of the same ordinal less than ω^2, then $E^{(a)} \equiv E^{(b)}$, but there are notations a, b of ω^2 such that $\mathrm{Id}^{(a)}$ and $\mathrm{Id}^{(b)}$ are incomparable. Moreover, there is no non-universal ceer which is an upper bound for all the ceers of the form $\mathrm{Id}^{(a)}$ where a is a notation for ω^2.

1. Introduction

Recently, there has been a renewed interest in studying equivalence relations on the set ω of natural numbers under the reducibility $\leq$, where $R \leq S$ if there exists a computable function f such that, for all $x, y \in \omega$,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

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The first systematic investigation of this reducibility goes back to Ershov (see e.g., [9]). More recent papers, with applications to computable model theory and computable algebra, include [7,11,10]. A natural and interesting particular case is provided by restriction of $\leq$ to computably enumerable equivalence relations (which will be abbreviated as *ceers*). The earliest paper fully dedicated to ceers (therein called *positive* equivalence relations) is [8], followed by other papers motivated by applications to logic, in view of the numerous examples of ceers naturally arising in logic: see for instance [3,4,13,14,17,19].

Let $\equiv$ be the equivalence relation given by $R \equiv S$ if $R \leq S$ and $S \leq R$. Then one can define, in the usual way, the degrees of equivalence relations (i.e., the $\equiv$-equivalence classes of equivalence relations), and in particular the poset $\mathcal{P}$ of degrees of ceers, which is bounded (see for instance [1]), i.e., with least element $\mathbf{0}$, consisting of just the trivial ceer, and greatest element $\mathbf{1}$: the ceers belonging to $\mathbf{1}$ are called *universal*. It is also worth recalling that this structure extends the structure of the 1-degrees of infinite c.e. sets: Given a set X define R_X to be the equivalence relation so that $x \mathrel{R_X} y$ if and only if $x = y$ or $x, y \in X$. It is not difficult to show (see for instance [1]) that if X, Y are infinite c.e. sets then $X \leq_1 Y$ if and only if $R_X \leq R_Y$; moreover if Y is c.e. and $Z \leq R_Y$ then $Z \equiv R_X$ for some c.e. set X. It turns out in this way that the interval of degrees of ceers $[\deg(\mathrm{Id}), \deg(R_K)]$ (where Id denotes the identity relation) is isomorphic to the interval of c.e. 1-degrees $[\mathbf{0}_1, \mathbf{0}'_1]$, where $\mathbf{0}_1$ is the 1-degree of any infinite and coinfinite decidable set, and $\mathbf{0}'_1$ is the 1-degree of the halting set K. This fact has been exploited in [1] to show that the first order theory of the poset of degrees of ceers is undecidable.

Gao and Gerdes [12] define a useful notion of jump of a ceer.

Definition 1.1. Given a ceer R, define

$$x \mathrel{R'} y \Leftrightarrow [x = y \text{ or } \varphi_x(x)\downarrow R\, \varphi_y(y)\downarrow] \,.$$

The ceer R' is called the *halting jump ceer of* R: in the following we simply call it the *jump* of R.

The main properties of the operation $R \mapsto R'$ are summarized in the following theorem:

Theorem 1.2. *[12,1] The following properties hold:*

(1) $R \leq R'$;
(2) $R \leq S$ *if and only if* $R' \leq S'$;
(3) *If* R *is not universal then* R' *is not universal;*
(4) *if* R *is not universal, then* $R < R'$.

Proof. Item (1) is [12, Proposition 8.3(1)]; item (2) is [12, Theorem 4]; item (3) is [12, Corollary 8.5(2)]; item (4) is [1, Theorem 4.3]. □

In particular we have a well-defined jump operation on degrees, given by $(\deg(R))' = \deg(R')$: this jump operation is an order embedding, and takes every degree to a bigger degree, except when it can not become strictly bigger, i.e., on the greatest element of $\mathcal{P}$. A degree of a ceer is a *jump degree* if it in the range of the jump operation on degrees.

Given equivalence relations R, S on ω, let $R \oplus S$ be defined by

$$x \mathrel{R \oplus S} y \Leftrightarrow \begin{cases} u \mathrel{R} v & \text{if } x = 2u \text{ and } y = 2v, \\ u \mathrel{S} v & \text{if } x = 2u + 1 \text{ and } y = 2v + 1. \end{cases}$$

Notice that $\oplus$ induces a well defined binary operations on degrees: if $\mathbf{a}$ and $\mathbf{b}$ are the degrees of R and S, respectively, then $\mathbf{a} \oplus \mathbf{b}$ is the degree of $R \oplus S$. Moreover $R \oplus S$ satisfies $R, S \leq R \oplus S$.