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Pure patterns of order 2

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We provide mutual elementary recursive order isomorphisms between classical ordinal notations, based on Skolem hulling, and notations from pure elementary patterns of resemblance of order 2, showing that the latter characterize the proof-theoretic ordinal 1^∞ of the fragment Π^1_1–CA_0 of second order number theory, or equivalently the set theory $\mathrm{KP}\ell_0$. As a corollary, we prove that Carlson's result on the well-quasi orderedness of respecting forests of order 2 implies transfinite induction up to the ordinal 1^∞. We expect that our approach will facilitate analysis of more powerful systems of patterns.

1. Introduction

Elementary patterns of resemblance were discovered and then systematically introduced by Timothy J. Carlson, [2–4], as an alternative approach to recursive systems of ordinal notations. Elementary patterns constitute the basic levels of Carlson's programmatic approach, *patterns of embeddings*, which is inspired by Gödel's program of using large cardinals to solve mathematical incompleteness, see e.g. [8,9]. It follows heuristics that axioms of infinity are in close correspondence with ordinal notations. The long-term goal of patterns of embeddings is therefore to find an ultra-finestructure for large cardinal axioms based on embeddings, thereby ultimately complementing inner model theory.

Patterns of resemblance, which instead of involving codings of embeddings, rely upon binary relations coding the property of elementary substructure of increasing complexity, are first steps to investigate patterns. Inspired by the notion of elementary substructure along ordinals as set-theoretic objects, ordinal

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notations in terms of elementary patterns intrinsically carry semantic content. However, Carlson made the intriguing observation that patterns have simple, finitely combinatorial characterizations called *respecting forests*.

The present article focuses on elementary patterns of order 2. Recalling from the introduction to [14], let $\mathcal{R}_2 = (\mathrm{Ord}; \le, \le_1, \le_2)$ be the structure of ordinals with standard linear ordering $\le$ and partial orderings $\le_1$ and $\le_2$, simultaneously defined by induction on β in

$$\alpha \le_i \beta :\Leftrightarrow (\alpha; \le, \le_1, \le_2) \preceq_{\Sigma_i} (\beta; \le, \le_1, \le_2)$$

where $\preceq_{\Sigma_i}$ is the usual notion of Σ_i-elementary substructure (without bounded quantification), see [1,3] for fundamentals and groundwork on elementary patterns of resemblance. Pure patterns of order 2 are the finite isomorphism types of $\mathcal{R}_2$. The *core* of $\mathcal{R}_2$ consists of the union of *isominimal realizations* of these patterns within $\mathcal{R}_2$, where a finite substructure of $\mathcal{R}_2$ is called isominimal, if it is pointwise minimal (with respect to increasing enumerations) among all substructures of $\mathcal{R}_2$ isomorphic to it, and where an isominimal substructure of $\mathcal{R}_2$ realizes a pattern P, if it is isomorphic to P. It is a basic observation, cf. [3], that the class of pure patterns of order 2 is contained in the class $\mathcal{RF}_2$ of *respecting forests of order* 2: finite structures P over the language $(\le_0, \le_1, \le_2)$ where $\le_0$ is a linear ordering and $\le_1, \le_2$ are forests such that $\le_2 \subseteq \le_1 \subseteq \le_0$ and $\le_{i+1}$ *respects* $\le_i$, i.e. $p \le_i q \le_i r \,\&\, p \le_{i+1} r$ implies $p \le_{i+1} q$ for all $p, q, r \in P$, for $i = 0, 1$.

In [7] we showed that every pattern has a cover below 1^∞, the least such ordinal. As outlined in [14], an order isomorphism (embedding) is a cover (covering, respectively) if it maintains the relations $\le_1$ and $\le_2$. The ordinal of $\mathrm{KP}\ell_0$, which axiomatizes limits of models of Kripke-Platek set theory with infinity, is therefore least such that there exist arbitrarily long finite $\le_2$-chains. Moreover, by determination of enumeration functions of (relativized) connectivity components of $\le_1$ and $\le_2$, we were able to describe these relations in terms of classical ordinal notations. The central observation in connection with this is that every ordinal below 1^∞ is the greatest element in a $\le_1$-chain in which $\le_1$- and $\le_2$-chains alternate, thus providing a formalism that allows precise localization of ordinals in terms of relativized connectivity components of the relations $\le_1$ and $\le_2$. We called such chains *tracking chains*, as they provide all $\le_2$-predecessors and the greatest $\le_1$-predecessors insofar as they exist.

In [14] we showed that the arithmetical characterization of the structure $\mathcal{R}_2$ up to the ordinal 1^∞, which we denoted as $\mathcal{C}_2$, is an elementary recursive structure. This guarantees the elementary recursiveness of the order isomorphisms between hull and pattern notations given here.

From these preparations we devise here an algorithm that assigns an isominimal realization within $\mathcal{C}_2$ to each respecting forest of order 2, thereby showing that each such respecting forest is in fact (up to isomorphism) a pure pattern of order 2. It turns out that isominimal realizations are pointwise minimal among all covers of the given forest. We therefore derive a method that calculates ordinals coded in pattern notations in terms of familiar hull notations, see [11].

The notion of closure introduced here further allows us to provide pattern notations for finite sets of ordinals below 1^∞. We are going to define an elementary recursive function that assigns describing patterns $P(\alpha)$ to ordinals $\alpha \in 1^\infty$. Recalling again from [14], a descriptive pattern for an ordinal α is a pattern, the isominimal realization of which contains α. Descriptive patterns are given in a way that makes a canonical choice for normal forms, since in contrast to the situation in $\mathcal{R}_1^+$, cf. [13,6], there is no unique notion of normal form in $\mathcal{R}_2$. The chosen normal forms are of least possible cardinality.

The mutual order isomorphisms between hull and pattern notations in the present article enable classification of a new independence result for $\mathrm{KP}\ell_0$, as was already announced [14]. We demonstrate that Carlson's result in [5], according to which the collection of respecting forests of order 2 is well-quasi-ordered with respect to coverings, cannot be proven in $\mathrm{KP}\ell_0$ or, equivalently, in the restriction $\Pi^1_1\mathrm{-CA}_0$ of second order number theory to Π^1_1-comprehension and set induction. On the other hand, we know that transfinite induction up to the ordinal 1^∞ of $\mathrm{KP}\ell_0$ suffices to show that every pattern is covered [7].