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Slow reflection

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A B S T R A C T

We describe a "slow" version of the hierarchy of uniform reflection principles over Peano Arithmetic (**PA**). These principles are unprovable in Peano Arithmetic (even when extended by usual reflection principles of lower complexity) and introduce a new provably total function. At the same time the consistency of **PA** plus slow reflection is provable in $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$. We deduce a conjecture of S.-D. Friedman, Rathjen and Weiermann: Transfinite iterations of slow consistency generate a hierarchy of precisely ε_0 stages between **PA** and $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ (where $\mathrm{Con}(\mathbf{PA})$ refers to the usual consistency statement).

The starting point for our work is the notion of slow consistency for (finite extensions of) Peano Arithmetic that has been introduced by Sy-David Friedman, Michael Rathjen and Andreas Weiermann in [10]. Up to an "index shift" (see below) it is defined as

$$\mathrm{Con}^\diamond(\mathbf{PA} + \varphi) :\equiv \forall_x (F_{\varepsilon_0}(x)\!\downarrow \to \mathrm{Con}(\mathbf{I\Sigma}_{x+1} + \varphi)). \tag{1}$$

This formula involves the function F_{ε_0} at stage ε_0 of the fast-growing hierarchy, due to Wainer and Schwichtenberg [22,24]. We work with the version used by Sommer [23]: Adopting his assignment of fundamental sequences $\lambda = \sup_{x\in\omega}\{\lambda\}(x)$ to limit ordinals $\lambda \leq \varepsilon_0$ (in particular $\{\varepsilon_0\}(x) = \omega_{x+1}$ is a tower of $x + 1$ exponentials with base ω) we define F_α by recursion on $\alpha \leq \varepsilon_0$, setting

$$F_0(x) := x + 1,$$

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$$F_{\alpha+1}(x) := F_\alpha^{x+1}(x),$$
$$F_\lambda(x) := F_{\{\lambda\}(x)}(x) \quad \text{for } \lambda \text{ a limit ordinal.}$$

To conceive of $\mathrm{Con}^\diamond(\mathbf{PA}+\varphi)$ as an arithmetic formula (of complexity Π_1), recall that ordinals below ε_0 can be represented via their Cantor normal forms. We adopt the efficient encoding of [23]. Building on this one can arithmetize the fast-growing hierarchy: Sommer in [23, Section 5.2] constructs a Δ_0-formula $F_\alpha(x)=y$ which defines the graphs of the functions F_α for $\alpha\leq\varepsilon_0$ (cf. [9, Equation 4] for the case $\alpha=\varepsilon_0$). Basic relations between these functions become provable in $\mathbf{I\Sigma}_1$. As usual $F_\alpha(x)\!\downarrow$ abbreviates $\exists_y F_\alpha(x)=y$. In addition, the formula $\mathrm{Con}^\diamond(\mathbf{PA}+\varphi)$ depends on a formula $\mathrm{Proof}_{\mathbf{I\Sigma}_x}(p,\varphi)$ which is Δ_1 in $\mathbf{I\Sigma}_1$ and arithmetizes the ternary relation "p is a proof of φ in the theory $\mathbf{I\Sigma}_x$". Here $\mathbf{I\Sigma}_x$ denotes the fragment of Peano Arithmetic in which induction is only available for Σ_x-formulas.

It is a classical result, due to Kreisel, Wainer and Schwichtenberg [15,22,24], that Peano Arithmetic does not prove $\forall_x F_{\varepsilon_0}(x)\!\downarrow$. This opens up the possibility that $\mathrm{Con}^\diamond$ is strictly weaker than the usual consistency statement. Friedman, Rathjen and Weiermann in [10] prove that this possibility materializes: Indeed, by [10, Section 4] finite iterations of slow consistency generate a strict hierarchy of ω theories that are stronger than Peano Arithmetic but bounded by the usual consistency statement $\mathrm{Con}(\mathbf{PA})$. It is conjectured in [10, Remark 4.4] that the same holds for a transfinite extension of the hierarchy up to any ordinal below ε_0. In the present paper we prove that this is the case: For an appropriate Π_1-formula $\mathrm{Con}^\diamond_\alpha(\mathbf{PA})$ in the variable α we have

$$\mathbf{PA}\subsetneq\cdots\subsetneq\mathbf{PA}+\mathrm{Con}^\diamond_\alpha(\mathbf{PA})\subsetneq\cdots\subsetneq\mathbf{PA}+\mathrm{Con}^\diamond_{\varepsilon_0}(\mathbf{PA})\equiv\mathbf{PA}+\mathrm{Con}(\mathbf{PA}).$$

As in [10, Theorem 3.1] this is also a strict hierarchy with respect to the interpretability ordering.

To prove the result about iterated slow consistency we introduce a notion of slow reflection which is interesting in its own right. As observed by Michael Rathjen in [20] slow consistency can be derived from a corresponding notion of slow provability, and indeed slow proof: A slow $\mathbf{PA}$-proof of a formula φ is a pair $\langle q, F_{\varepsilon_0}(n)\rangle$ such that q is a usual proof of φ in the fragment $\mathbf{I\Sigma}_{n+1}$. Writing π_i for the projections of the Cantor pairing function this amounts to the formula

$$\mathrm{Proof}^\diamond_{\mathbf{PA}}(p,\varphi):\equiv\exists_x(\mathrm{Proof}_{\mathbf{I\Sigma}_{x+1}}(\pi_1(p),\varphi)\wedge F_{\varepsilon_0}(x)=\pi_2(p))$$

which is Δ_1 in $\mathbf{I\Sigma}_1$ (cf. [9, Definition 2.1]). Slow provability is then defined as

$$\mathrm{Pr}^\diamond_{\mathbf{PA}}(\varphi):\equiv\exists_p\ \mathrm{Proof}^\diamond_{\mathbf{PA}}(p,\varphi).$$

Michael Rathjen shows in [20] that slow provability realizes Gödel–Löb provability logic (see also Lemma 3.10 below). It is easy to see that we have

$$\mathbf{I\Sigma}_1\vdash\forall_\psi(\mathrm{Pr}^\diamond_{\mathbf{PA}}(\psi)\leftrightarrow\exists_x\,(F_{\varepsilon_0}(x)\!\downarrow\wedge\,\mathrm{Pr}_{\mathbf{I\Sigma}_{x+1}}(\psi))) \tag{2}$$

and then

$$\mathbf{I\Sigma}_1\vdash\mathrm{Con}^\diamond(\mathbf{PA}+\varphi)\leftrightarrow\neg\,\mathrm{Pr}^\diamond_{\mathbf{PA}}(\neg\varphi).$$

Given a notion of provability one can consider the corresponding reflection principles. We will mainly be concerned with uniform reflection. Using Feferman's dot notation, slow (uniform) reflection for the formula $\varphi\equiv\varphi(x_1,\dots,x_k)$ is defined as

$$\mathrm{RFN}^\diamond_{\mathbf{PA}}(\varphi):\equiv\forall_{x_1,\dots,x_k}(\mathrm{Pr}^\diamond_{\mathbf{PA}}(\varphi(\dot{x}_1,\dots,\dot{x}_k))\to\varphi(x_1,\dots,x_k)).$$

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