Journal of Applied Logic 25 (2017) 23–46

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

A more general general proof theory

Heinrich Wansing

Department of Philosophy II, Ruhr-University Bochum, Germany

ARTICLE INFO

Article history:
Received 18 July 2016
Accepted 10 November 2016

Keywords:
Proofs
Dual proofs
Negation
Co-implication
2Int
Bi-intuitionistic logic
Dual intuitionistic logic
Normal form
Assertion
Denial
Compositionality
Bilateralism

ABSTRACT

In this paper it is suggested to generalize our understanding of general (structural) proof theory and to consider it as a general theory of two kinds of derivations, namely proofs and dual proofs. The proposal is substantiated by (i) considerations on assertion, denial, and bi-lateralism, (ii) remarks on compositionality in proof-theoretic semantics, and (iii) comments on falsification and co-implication. The main formal result of the paper is a normal form theorem for the natural deduction proof system N2Int of the bi-intuitionistic logic 2Int. The proof makes use of the faithful embedding of 2Int into intuitionistic logic with respect to validity and shows that conversions of dual proofs can be sidestepped.

1. Introduction

In the early 1970s Dag Prawitz introduced general proof theory as "a study of proofs in their own right where one is interested in general questions about the nature and structure of proofs" [16, p. 66], and in his seminal paper on "Ideas and results in proof theory" [15], Prawitz listed what he considered to be obvious topics in general proof theory:

2.1. The basic question of defining the notion of proof, including the question of the distinction between different kinds of proofs such as constructive proofs and classical proofs.

2.2. Investigation of the structure of (different kinds of) proofs, including e.g. questions concerning the existence of certain normal forms.

2.3. The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations

E-mail address: Heinrich.Wansing@rub.de.

http://dx.doi.org/10.1016/j.jal.2017.01.002
1570-8683/

represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.

2.4. Applications of insights about the structure of proofs to other logical questions that are not formulated in terms of the notion of proof.

In this paper, I will suggest to broaden Prawitz's understanding of general proof theory. The idea is to consider in addition to verifications also falsifications, so as to obtain a theory of proofs and what I call "dual proofs." General proof theory would thus be a study of proofs and dual proofs in their own right where one is interested in general questions about the nature and structure of proofs and dual proofs.

The topic can be approached from different perspectives. I shall consider (i) the speech acts of assertion and denial, (ii) the problem of compositionality in proof-theoretic semantics, and (iii) the idea of supplementing intuitionistic implication with an operation of co-implication.

1.1. Assertion and denial

The speech acts of assertion and denial are usually seen to correspond to certain inner actions and to certain propositional attitudes. I will follow Ripley [23], who uses the terms "'deny' and 'denial' exclusively to pick out a certain type of speech act: the sort someone is engaged in when they deny something" and who similarly uses "'reject' and 'rejection' to pick out a certain type of attitude: the sort someone has to a content when they reject it." Timothy Williamson [37, p. 10] explains that "we can regard assertion as the verbal counterpart of judgement and judgement as the occurrent form of belief." If assertion is the verbal expression of the attitude of belief towards a propositional content, then denial is the verbal expression of the attitude of rejection towards a propositional content. The attitude verbalized by a denial may be understood as disbelief or, perhaps, a weaker form of rejection, as for example in [19].

In the context of general proof theory, the speech acts of assertion and denial are considered in Per Martin-Löf's 1983 lectures "On the Meanings of the Logical Constants and the Justifications of the Logical Laws" [11], where he takes up Gottlob Frege's understanding of inferences as transitions between judgements or assertions, which is carefully explained in [28], see also [29]. A presentation of the Fregean conception of inference can also be found in a recent paper by Prawitz [18, p. 67 f.]:

> [A] reflective inference contains at least a number of assertions or judgements made in the belief that one of them, the conclusion, say $\mathcal{B}$, is supported by the other ones, the premises, say $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$. An inference in the course of an argument or proof is not an assertion or judgement to the effect that $\mathcal{B}$ "follows" from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, but is first of all a transition from some assertions (or judgements) to another one. ...
>
> This is how Frege saw an inference, as a transition between assertions or judgements. To make an assertion is to use a declarative sentence A with assertive force, which we may indicate by writing $\vdash A$, using the Fregean assertion sign. We may also say with Frege that a sentence A expresses a thought or proposition p, while $\vdash A$, the assertion of A, is an act in which p is judged to be true, not to be confused with ascribing to p the property of being true.

This is only a first characterization of inferences, and Prawitz develops a more general conception by considering, unlike Frege, assertions under assumptions, written as a sequent $A_1, A_2, \ldots, A_n \vdash B$, open assertions (that is, assertions of open sentences), and inference figures, namely structures

$$\frac{\Gamma_1 \vdash A_1 \quad \Gamma_2 \vdash A_2 \ldots \Gamma_n \vdash A_n}{\Delta \vdash B}$$