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ON ENTIRE SOLUTIONS OF SOME TYPE OF NONLINEAR DIFFERENCE EQUATIONS*

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Abstract In this article, the existence of finite order entire solutions of nonlinear difference equations $f^n + P_d(z, f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$ are studied, where $n \geq 2$ is an integer, $P_d(z, f)$ is a difference polynomial in f of degree $d(\leq n-2)$, p_1, p_2 are small meromorphic functions of e^z, and α_1, α_2 are nonzero constants. Some necessary conditions are given to guarantee that the above equation has an entire solution of finite order. As its applications, we also find some type of nonlinear difference equations having no finite order entire solutions.

Key words Nevanlinna theory; difference polynomial; difference equation; entire solution

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1 Introduction and Main Results

Let f be a meromorphic function in the complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory of meromorphic functions, such as $m(r, f), N(r, f), T(r, f)$ etc.; see, for example, [1, 2]. The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set E of r of finite logarithmic measure. A meromorphic function α is said to be a small function of f provided that $T(r, \alpha) = S(r, f)$. In general, a difference polynomial, or a differential-difference polynomial in f is defined to be a finite sum of difference products of f and its shifts $f(z + c_j), (c_j \in \mathbb{C}, j \in I)$, or of products of f, derivatives of f, and of their shifts, with small meromorphic coefficients, where I is a finite index set.

It is an important and difficult problem for complex differential equations to prove the existence of their solutions. In [3], Yang and Li pointed out that the differential equation $4f^3(z) + 3f''(z) = -\sin 3z$ has exactly three nonconstant entire solutions $f_1(z) = \sin z, f_2(z) =$

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$\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$, and $f_3(z)=-\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$. Furthermore, the existence of entire solutions of the more general differential equation

$$f^n(z)+Q_d(z,f)=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}, \tag{1.1}$$

where $Q_d(z,f)$ is a differential polynomial in f of degree d, have attracted many interests (see [4–7] etc.). In [4], Li and Yang proved the following result.

Theorem A ([4]) Let $n\geq 4$ be an integer and $d\leq n-3$. If p_1,p_2 are nonzero polynomials, and α_1,α_2 are nonzero constants such that $\frac{\alpha_1}{\alpha_2}$ is not rational, then equation (1.1) does not have any transcendental entire solutions.

When weakening the restriction on d, Li [5] proved the following result.

Theorem B ([5]) Let $n\geq 2$ be an integer, $d\leq n-2$, p_1,p_2 be nonzero small functions of e^z, and let α_1,α_2 be real numbers. If $\alpha_1<0<\alpha_2$ and equation (1.1) has a transcendental entire solution f, then $\alpha_1+\alpha_2=0$ and $f=c_1\beta_1e^{\frac{\alpha_1 z}{n}}+c_2\beta_2e^{\frac{\alpha_2 z}{n}}$, where c_1,c_2 are constants and $\beta_j^n=p_j(j=1,2)$.

Recently, replacing the differential polynomial $Q_d(z,f)$ in equation (1.1) by a difference, or differential-difference polynomial $P_d(z,f)$, many authors [8–10] investigated the existence of entire solutions of the difference, or differential-difference equations

$$f^n(z)+P_d(z,f)=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}, \tag{1.2}$$

where $P_d(z,f)$ is a difference, or differential-difference polynomial in f of degree d. In [9], Zhang and Liao obtained a counterpart of Theorem A for entire solutions of finite order of equation (1.2).

Theorem C ([9]) Let $n\geq 4$ be an integer, and $P_d(z,f)$ denote an algebraic differential-difference polynomial in f of degree $d\leq n-3$. If p_1,p_2 are two nonzero polynomials, and α_1,α_2 are two nonzero constants with $\frac{\alpha_1}{\alpha_2}\neq\left(\frac{d}{n}\right)^{\pm1},1$, then equation (1.2) does not have any transcendental entire solution of finite order.

It is natural to ask whether equation (1.2) has any entire solution of finite order if we weaken the restriction $d\leq n-3$ in Theorem C. To this end, we prove the following result.

Theorem 1.1 Let $n\geq 2$ be an integer, $P_d(z,f)$ be a difference polynomial in f of degree $d\leq n-2$ such that $P_d(z,0)\not\equiv 0$, p_1,p_2 be nonzero small functions of e^z, and let α_1,α_2 be constants. If $\frac{\alpha_1}{\alpha_2}<0$, and equation (1.2) has an entire solution f of finite order, then $\alpha_1+\alpha_2=0$ and $f=\gamma_1e^{\frac{\alpha_1 z}{n}}+\gamma_2e^{\frac{\alpha_2 z}{n}}$, where $\gamma_j(j=1,2)$ are small functions of f such that $\gamma_j^n=p_j(j=1,2)$.

Remark 1.1 The below proof of Theorem 1.1 shows that the result of Theorem 1.1 also holds for the case $P_d(z,f)$ being a differential-difference polynomial in f of degree $d\leq n-2$.

Remark 1.2 There exist nonlinear difference (or differential-difference) equations satisfying Theorem 1.1. For example, the difference equation

$$f^5(z)+5f^3\left(z+\frac{\pi \mathrm{i}}{\lambda}\right)-f^2\left(z+\frac{\pi \mathrm{i}}{\lambda}\right)+f\left(z+\frac{5\pi \mathrm{i}}{2\lambda}\right)f\left(z-\frac{\pi \mathrm{i}}{2\lambda}\right)+5f(z)+4=e^{5\lambda z}+e^{-5\lambda z}$$

has an entire solution $f(z)=e^{\lambda z}+e^{-\lambda z}$. The differential-difference equation

$$f^4(z)+4e^{\frac{\pi \mathrm{i}}{4}}f\left(z+\frac{3\pi}{8}\right)f'\left(z+\frac{\pi}{8}\right)-(6+4\sqrt{2})\mathrm{i}=2\mathrm{i}\sin 4z$$

has an entire solution $f(z)=e^{\mathrm{i}z}+e^{\frac{\pi \mathrm{i}}{4}}e^{-\mathrm{i}z}$.