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# MULTIPLICITY OF SOLUTIONS OF WEIGHTED $(p, q)$-LAPLACIAN WITH SMALL SOURCE*

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**Abstract** In this article, we study the existence of infinitely many solutions to the degenerate quasilinear elliptic system

$$\begin{aligned} -\mathrm{div}(h_1(x)|\nabla u|^{p-2}\nabla u) &= \mathrm{d}(x)|u|^{r-2}u + G_u(x,u,v) \qquad \text{in } \Omega, \\ -\mathrm{div}(h_2(x)|\nabla v|^{q-2}\nabla v) &= f(x)|v|^{s-2}v + G_v(x,u,v) \qquad \text{in } \Omega, \\ u = v &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \ge 2$, $1 < r < p < \infty$, $1 < s < q < \infty$; $h_1(x)$ and $h_2(x)$ are allowed to have "essential" zeroes at some points in $\Omega$; $\mathrm{d}(x)|u|^{r-2}u$ and $f(x)|v|^{s-2}v$ are small sources with $G_u(x,u,v)$, $G_v(x,u,v)$ being their high-order perturbations with respect to $(u, v)$ near the origin, respectively.

**Key words** Weighted $(p, q)$-Laplacian, small sources, multiplicity

**2010 MR Subject Classification** 35J70, 35J50

## 1 Introduction

The purpose of this article is to study the multiplicity of solutions for the system with small sources:

$$\begin{cases} -\mathrm{div}(h_1(x)|\nabla u|^{p-2}\nabla u) = \mathrm{d}(x)|u|^{r-2}u + G_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(h_2(x)|\nabla v|^{q-2}\nabla v) = f(x)|v|^{s-2}v + G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < r < p < \infty$, $1 < s < q < \infty$; $h_1(x)$ and $h_2(x)$ are allowed to have "essential" zeroes at some points in $\Omega$; $d(x)$ and $f(x)$ can be very small, in particular, small supports and sign changing for $d(x)$ and $f(x)$ are permitted and the terms $G_u(x,u,v)$ and $G_v(x,u,v)$ will be considered as high-order perturbations of the small sources $\mathrm{d}(x)|u|^{r-2}u$ and $f(x)|v|^{s-2}v$ with respect to $(u,v)$ near the origin respectively.

For the semilinear case of single equation

$$\begin{cases} -\Delta u = \lambda|u|^{r-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $r \in (1,2)$ and $\lambda > 0$, existence of infinitely many solutions has attracted much attention and has been extensively studied in the last three decades. For example, in [1] Ambrosetti-Badiale obtained infinitely many solutions of (1.2) with negative energy when $g(x,u) \equiv 0$, using a dual variational formulation. Ambrosetti-Brezis-Cerami [2] and Garcia-Peral [3] proved that (1.2) has infinitely many solutions with negative energy provided that $g(x,u) = |u|^{m-2}u$, $m \in (2, 2^*]$, where $2^* = 2N/(N-2)$ for $N \geq 3$ and $2^* = \infty$ for $N = 1, 2$, and $0 < \lambda < \lambda^*$ for some finite $\lambda^*$. For $2 < m < 2^*$, Bartsch-Willem [4] removed the restriction on $\lambda$ and obtained infinitely many solutions under the assumptions that $g(x,u) = \mu|u|^{m-2}u$, $m \in (2, 2^*)$, $\mu \in \mathbb{R}$, and $\lambda > 0$. It should be noted that in all quoted articles above, the global property of $g(x,u)$ for $u$ large was used in an essential way to derive multiplicity results of solutions with negative energy. It was Wang [5] who first observed that existence of infinitely many solutions of (1.2) with negative energy relies only on local behavior of the equation and assumptions on $g(x,u)$ only for small $u$ are required. More precisely, he proved that if $1 < r < 2$, $g \in C(\overline{\Omega} \times (-\delta,\delta), \mathbb{R})$ for some $\delta > 0$, $g$ is odd in $u$ and $g(x,u) = o(|u|^{r-1})$ as $|u| \to 0$ uniformly in $x \in \Omega$, then for all $\lambda > 0$, (1.2) has a sequence of weak solutions with negative energy, thus improving all the previous results. It is worth pointing out that positivity $\lambda > 0$ plays a crucial role in his argument. Recently, Guo [6] and Jing-Liu [7] considered the following problem

$$\begin{cases} -\Delta u = \mathrm{d}(x)|u|^{r-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $d \in C(\overline{\Omega})$ is allowed to change sign, more exactly,

$$\Omega_d^+ = \{x \in \Omega | \mathrm{d}(x) > 0\} \neq \varnothing. \tag{1.4}$$

In [6], Guo proved that if $1 < r < 2$, (1.4) holds, $g \in C(\overline{\Omega} \times (-\delta,\delta), \mathbb{R})$ for some $\delta > 0$, $g$ is odd in $u$ and $g(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \Omega$, then (1.3) has a sequence of nontrivial solutions whose $L^\infty$-norms converge to zero. Jing-Liu [7] removed the oddness of the perturbation term $g$, and assumed that $g \in C(\overline{\Omega} \times (-\delta,\delta), \mathbb{R})$ for some $\delta > 0$, $g(x,u) = o(|u|^{\tau-1})$ as $|u| \to 0$ uniformly in $x \in \Omega$, where $2 + N(2-r)/r < \tau \leq 2^*$ for $N \geq 3$ and $2 + N(2-r)/r < \tau$ for $N = 1, 2$. Chung [8] further showed the existence of infinitely many solutions of the system (1.1) with linear principal parts, that is,

$$p = q = 2,$$