Available online at www.sciencedirect.com

ScienceDirect

Acta Mathematica Scientia 2018,**38B**(2):441–449

Acta Mathematica Scientia 数学物理学报

http://actams.wipm.ac.cn

SHARP BOUNDS FOR HARDY OPERATORS ON PRODUCT SPACES*

Mingquan WEI (魏明权)†

School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

E-mail: weimingquan11@mails.ucas.ac.cn

Dunyan YAN (燕敦验)

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

E-mail: ydunyan@ucas.ac.cn

Abstract In this article, we obtain the sharp bounds from $L^{\mathbf{P}}(\mathbb{G}^n)$ to the space $wL^{\mathbf{P}}(\mathbb{G}^n)$ for Hardy operators on product spaces. More generally, the precise norms of Hardy operators on product spaces from $L^{\mathbf{P}}(\mathbb{G}^n)$ to the space $L^{\mathbf{P_I}}(\mathbb{G}^n)$ are obtained.

Key words Hardy operators; product spaces; $wL^{\mathbf{P}}(\mathbb{G}^n)$; $L^{\mathbf{P_I}}(\mathbb{G}^n)$; norm

2010 MR Subject Classification 42B20; 42B35

1 Introduction

Let f be a non-negative integrable function on $\mathbb{G}$. The classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \tag{1.1}$$

for $x \in \mathbb{G}$, where $\mathbb{G} = (0, +\infty)$.

The following Theorem A, because of Hardy [1], is well-known.

Theorem A If f is a non-negative measurable function on $\mathbb{G}$ and $1 < p < \infty$, then

$$\|Hf\|_{L^p(\mathbb{G})} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{G})}$$

holds, and the constant $\frac{p}{p-1}$ is sharp.

In 1976, Faris [2] first gave a definition of Hardy operator in n-dimensional case. In 1995, Christ and Grafakos [3] gave its equivalent version of n-dimensional Hardy operator

$$\mathcal{H}_n f(x) := \frac{1}{\Omega_n|x|^n}\int_{|y|<|x|} f(y)\mathrm{d}y,\ x \in \mathbb{R}^n\backslash\{0\}, \tag{1.2}$$

where f is a non-negative measurable function on $\mathbb{R}^n$ and $\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$. The norm of $\mathcal{H}_n$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, was evaluated in [3] and found to be

*Received June 7, 2016; revised June 23, 2017. The work is supported by NSFC (11471309, 11271162, and 11561062), Project of Henan Provincial Department of Education (18A110028), the Nanhu Scholar Program for Young Scholars of XYNU, and Doctoral Scientific Research Startup Fund of Xinyang Normal University (2016).

†Corresponding author

equal to the one dimensional Hardy operator, that is,

$$\|\mathcal{H}_n\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \frac{p}{p-1}. \tag{1.3}$$

For the case of n-dimensional product spaces, the Hardy operator is defined by

$$H_n f(x) := \frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{1.4}$$

for $x=(x_1,x_2,\cdots,x_n)\in\mathbb{G}^n$, where $\mathbb{G}^n=(0,+\infty)\times\cdots\times(0,+\infty)$ is a product space and f is a non-negative measurable function on $\mathbb{G}^n$. In 1992, Pachpatte [4] investigated the operator defined by (1.4) and obtained the following Theorem B.

Theorem B If f is a non-negative measurable function on $\mathbb{G}^n$ and $1<p<\infty$, then

$$\|H_n f\|_{L^p(\mathbb{G}^n)} \le \left(\frac{p}{p-1}\right)^n \|f\|_{L^p(\mathbb{G}^n)}, \tag{1.5}$$

where the constant $(\frac{p}{p-1})^n$ is best possible.

It is often important to obtain sharp norm estimates for Hardy-type integral inequalities. Readers can refer to [5] to get some earlier development of Hardy operator. There are also many articles dealing with such inequalities such as [6, 7, 21, 22].

As $|\mathcal{H}_n f|\le Mf$ and the centered Hardy-Littlewood maximal operator is of weak type $(1,1)$, it is obtained that $\mathcal{H}_n$ is also of weak type $(1,1)$. A. Melas [8] found the exact value of the best possible constant $C=1.5675208\cdots$ for the weak-type $(1,1)$ inequality of one-dimensional centered Hardy-Littlewood maximal operator. However, the best constant in the weak type $(1,1)$ inequality for certain centered maximal operators in $\mathbb{R}^n (n\ge 2)$ is still open (see [9]).

In 2011, Fayou Zhao et al [20] got the weak-type (p,p) bound of n-dimensional Hardy operator, which is stated as follows.

Theorem C For $1\le p\le\infty$, the following inequality

$$\|\mathcal{H}_n f\|_{L^{p,\infty}(\mathbb{R}^n)} \le 1\cdot\|f\|_{L^p(\mathbb{R}^n)} \tag{1.6}$$

holds. Moreover,

$$\|\mathcal{H}_n\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} = 1, \tag{1.7}$$

where the bound 1 is best.

Unfortunately, the operator H_n is even not of weak type $(1,1)$ when $n\ge 2$ (In Section 2, we will give an example). In order to study the general weak $(1,1)$ type inequality for H_n, we shall consider another space instead of $L^1(\mathbb{G}^n)$. The refined definitions of mixed-norm space will be presented in Section 2.

2 Preliminaries

Before we prove the main results, some useful lemmas and definitions will first be given.

Lemma 2.1 The operator H_n, $n\ge 2$, defined by (1.4) fails to be of weak type of $(1,1)$.

Proof We merely consider the case $n=2$, and the same is true for $n\ge 3$.

Let $f(t)=\chi_{(0,1]}(t)$, where χ_A is the characteristic function of the set A. Define a binary function on $\mathbb{G}^2$ as $F(x,y)=f(x)f(y)$ and then F is our desired function. For more details of the proof, readers can refer to [10] by Di Wu. □