Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2018,**38B**(2):497–518
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF CONCAVE-CONVEX ELLIPTIC EQUATIONS WITH CRITICAL GROWTH*

Jiafeng LIAO (廖家锋) *Yang PU* (蒲洋)
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
School of Mathematics and Information, China West Normal University, Nanchong 637002, China
E-mail: *liaojiafeng@163.com; 172431794@qq.com*

Chunlei TANG (唐春雷)†
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
E-mail: *tangcl@swu.edu.cn*

Abstract In this article, the following concave and convex nonlinearities elliptic equations involving critical growth is considered,

$$\begin{cases} -\Delta u = g(x)|u|^{2^*-2}u + \lambda f(x)|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded domain with smooth boundary, $1 < q < 2, \lambda > 0$. $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $f \in L^{\frac{2^*}{2^*-q}}(\Omega)$ is nonzero and nonnegative, and $g \in C(\overline{\Omega})$ is a positive function with k local maximum points. By the Nehari method and variational method, $k+1$ positive solutions are obtained. Our results complement and optimize the previous work by Lin [MR2870946, Nonlinear Anal. 75(2012) 2660-2671].

Key words Semilinear elliptic equations; critical growth; positive solutions; Nehari method; variational method

2010 MR Subject Classification 35B09; 35B33; 35J20

1 Introduction and Main Result

In this article, we study the following concave-convex elliptic equations involving critical Sobolev exponent

$$\begin{cases} -\Delta u = g(x)|u|^{2^*-2}u + \lambda f(x)|u|^{q-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

* Received August 3, 2016; revised October 17, 2017. Supported by National Natural Science Foundation of China(11471267), the Doctoral Scientific Research Funds of China West Normal University (15D006 and 16E014), Meritocracy Research Funds of China West Normal University (17YC383), and Natural Science Foundation of Education of Guizhou Province (KY[2016]046).

†Corresponding author

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded domain with smooth boundary, $1 < q < 2, \lambda > 0$. $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for every $p \in [1, \frac{2N}{N-2}]$, where $H_0^1(\Omega)$ is a Sobolev space equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. The coefficient function $f \in L^r(\Omega)$ is nonzero and nonnegative, where $r = \frac{2^*}{2^*-q}$. And $g \in C(\overline{\Omega})$ is a positive function.

More precisely, we say that a function $u \in H_0^1(\Omega)$ is called a weak solution of problem (1.1), if for all $\varphi \in H_0^1(\Omega)$, there holds

$$\int_\Omega (\nabla u, \nabla \varphi)\mathrm{d}x - \int_\Omega g(x)(u^+)^{2^*-1}\varphi \mathrm{d}x - \lambda \int_\Omega f(x)(u^+)^{q-1}\varphi \mathrm{d}x = 0, \tag{1.2}$$

where $u^+ = \max\{u, 0\}$.

It is well known that the pioneer work is Brézis and Nirenberg [4], that is, the existence of positive solutions of semilinear elliptic equations involving critical exponent is related to the dimension of space. After that, semilinear elliptic problems with critical exponent were extensively considered (see [1, 2, 5, 7–30, 33, 35–37, 39–42]). Particularly, Ambrosetti, Brézis, and Cerami [2] studied the following problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < q < 1 < p \leq 2^*-1$. They obtained the classic results by the sub-supersolution method and variational method, that is, there exists $\lambda_0 > 0$ such that problem (1.3) has at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$, and no positive solution for $\lambda > \lambda_0$. After that, many authors considered the concave-convex-type elliptic problems (see [1, 9, 14, 20, 21, 24–28, 35, 40]). Particularly, Korman in [22] considered the exact number of positive solutions for problem (1.3) in the unit ball $B \subset \mathbb{R}^N (N \geq 3)$. By the bifurcation theory, it is obtained that there exists a critical number $\tilde{\lambda} > 0$ such that problem (1.3) has two positive solutions for $\lambda \in (0, \tilde{\lambda})$, exactly one positive solution for $\lambda = \tilde{\lambda}$, and no positive solution for $\lambda > \tilde{\lambda}$. As any positive solution to problem (1.3) in the unit ball B is radial, Tang in [35] proved this result of [22] by an ordinary differential equation method. In [1], Ambrosetti, Azorer, and Peral studied problem (1.3) in $\mathbb{R}^N (N \geq 3)$, that is,

$$-\Delta u = u^{2^*-1} + \lambda h(x)u^q, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $0 < q < 1$, $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. When $h \geq 0, h \not\equiv 0$, and $\lambda > 0$ small enough, they obtained two positive solutions. Moreover, [9, 14] and [20] considered the multiplicity of positive solutions for concave-convex p-Laplacian problems with critical Sobolev exponent.

Recently, Lin considered problem (1.1) under the following conditions in [28].

(h_1) $f, g \in C(\overline{\Omega})$, $f \geq 0, f \not\equiv 0$, and $g > 0$.

(h_2) There exist k points $a^1, a^2, \cdots, a^k$ in Ω such that

$$g(a^i) = \max_{x \in \Omega} g(x) = 1 \quad \text{for } 1 \leq i \leq k,$$

and for some $\sigma > N$ such that $g(x) - g(a^i) = O(|x - a^i|^\sigma)$ as $x \to a^i$ uniformly in i.

(h_3) Choosing $r_0 > 0$ such that

$$\overline{B_{r_0}(a^i)} \cap \overline{B_{r_0}(a^j)} = \emptyset \quad \text{for } i \neq j \text{ and } 1 \leq i, j \leq k,$$