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TWO DIMENSIONAL MELLIN TRANSFORM IN QUANTUM CALCULUS*

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Abstract In this article, we introduce the two dimensional Mellin transform $M_{\bar{q}}(f)(s,t)$, give some properties, establish the Paley-Wiener theorem and Plancherel formula, present the Hausdorff-Young inequality, and find several applications for the two dimensional Mellin transform.

Key words Quantum calculus; Two dimensional Mellin transform; q-Double integral

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1 Introduction

It is well known that the integral transforms are very important in the areas of science and engineering, and they attracted the attention of many researchers (see [1–5]). Two of the most frequently used formulas in the area of integral transforms are the classical Mellin transform and the corresponding formal inversion formula; they were successfully applied in the theory of differential equations, plain harmonic problems in special domains, elasticity mechanics, special functions, summing series, and calculating integrals.

In 1854–1933, Hjalmar Mellin defined the Mellin transform of a suitable function f over $(0,\infty)$ as

$$M(f)(s)=\int_0^{\infty}x^{s-1}f(x)\mathrm{d}x. \tag{1.1}$$

In 2006, A.Fitouhi et al [5] studied the q-analogue of the Mellin transform and its inversion given, respectively, by

$$M_q(f)(s)=\int_0^{\infty}x^{s-1}f(x)\mathrm{d}_qx,\ \forall x\in\mathbb{R}_{q,+}, \tag{1.2}$$

and

$$f(x)=\frac{\log(q)}{2\mathrm{i}\pi(1-q)}\int_{c-\frac{\mathrm{i}\pi}{\log(q)}}^{c+\frac{\mathrm{i}\pi}{\log(q)}}M_q(f)(s)x^{-s}\mathrm{d}s,\quad c\in(\alpha_{q,f},\beta_{q,g}),$$

where $\mathbb{R}_{q,+}:=\{q^n,\ n\in\mathbb{Z}\}$.

As a generalization of the Mellin transform, the two-dimensional Mellin transform is defined by [2, 13]

$$M[f(x,y)](s,t)=\int_0^{\infty}\int_0^{\infty}f(x,y)x^{s-1}y^{t-1}\mathrm{d}x\mathrm{d}y; \tag{1.3}$$

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the inversion formula for the two-dimensional Mellin transform is given by the following relation

$$f(x,y)=-\frac{1}{4\pi^2}\int_{c_1-\mathrm{i}\infty}^{c_1+\mathrm{i}\infty}\int_{c_2-\mathrm{i}\infty}^{c_2+\mathrm{i}\infty}M(f)(s,t)x^{-s}y^{-t}\mathrm{d}s\mathrm{d}t. \tag{1.4}$$

The two-dimensional Mellin convolution product of the functions f and g is defined by

$$f**_M g(x,y)=\int_0^\infty\int_0^\infty u^{-1}v^{-1}f\Big(\frac{x}{u},\frac{y}{v}\Big)g(u,v)\mathrm{d}u\mathrm{d}v.$$

In [8], using two parameters of deformation q_1 and q_2, Haran et al gave the definition of an analogue of the two-dimensional Mellin transform by

$$M_{q_1,q_2}(f)(s,t):=\int_0^\infty\int_0^\infty f(x,y)x^{s-1}y^{t-1}\mathrm{d}_{q_1}x\mathrm{d}_{q_2}y. \tag{1.5}$$

The aim of this article is devoted first to study the analogue of the Mellin transform $M_{q_1,q_2}(f)$ (1.5) and second to discuss its properties and to give its inversion formula which is an analogue of (1.4). Furthermore, we define the convolution product. And finally as applications, we prove an analogue of the Titchmarsch theorem.

This article is organized as follows: In Section 2, we present some preliminary results and notations that will be useful in the sequel. In Section 3, we introduce the $\tilde{q}$-analogue of the two-dimensional Mellin transform, give some properties, and prove the inversion formula of the $\tilde{q}$-two dimensional Mellin transform. In Section 4, we study the convolution product and give some relations of the $\tilde{q}$-analogue of the two-dimensional Mellin transform. In Section 5, we give some applications. Finally, in Section 6, we establish Paley-Wiener theorems for the modified $\tilde{q}$-two-dimentional Mellin transform.

2 Basic Definitions

For the convenience of the reader, in this section we provide a summary of the mathematical notations and definitions used in this article (see [6, 9, 10]).

For $q\in(0,1)$ and $a\in\mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad n=1,2,\cdots. \tag{2.1}$$

$$(a;q)_\infty=\lim_{n\to+\infty}(a;q)_n=\prod_{k=0}^{\infty}(1-aq^k). \tag{2.2}$$

We also denote

$$[a]_q=\frac{1-q^a}{1-q},\qquad [n]_q!=\frac{(q;q)_n}{(1-q)^n},\qquad n\in\mathbb{N}.$$

The q-derivatives D_qf and D_q^+f of a function f are given by [10]:

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x},\qquad (D_q^+f)(x)=\frac{f(q^{-1}x)-f(x)}{(1-q)x},\qquad \text{if}\quad x\neq 0, \tag{2.3}$$

$(D_qf)(0)=f'(0)$ and $(D_q^+f)(0)=q^{-1}f'(0)$ provided $f'(0)$ exists.

If f is differentiable, then $(D_qf)(x)$ and $(D_q^+f)(x)$ tend to $f'(x)$ as q tends to 1. For $n\in\mathbb{N}$, we note

$$D_q^1=D_q,\qquad D_q^n=D_q(D_q^{n-1}),$$