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NUMERICAL SIMULATIONS FOR A VARIABLE ORDER FRACTIONAL CABLE EQUATION*

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Abstract In this article, Crank-Nicolson method is used to study the variable order fractional cable equation. The variable order fractional derivatives are described in the Riemann-Liouville and the Grünwald-Letnikov sense. The stability analysis of the proposed technique is discussed. Numerical results are provided and compared with exact solutions to show the accuracy of the proposed technique.

Key words Crank-Nicolson method; variable order fractional cable equation; stability analysis

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1 Introduction

The cable equation has been recently treated by numerous of authors and is found to be a useful approach for modeling neuronal dynamics. The cable equation can be deduced from the Nernst-Planck equation for electrodiffusion in smooth homogeneous cylinders and it was investigated for modeling the anomalous diffusion in spiny neuronal dendrites. Henry et al [4] derived a fractional cable equation from the fractional Nernst-Planck equations to model anomalous electrodiffusion of ions in spiny dendrites which is similar to the traditional cable equation except that the order of derivative with respect to the time or space is fractional.

Fractional calculus is a mathematical branch investigating the properties of integrals and derivatives of non-integer orders. The applications of fractional calculus occur in several fields, such as physics, engineering, fluid flow, viscoelasticity, and control theory of dynamical systems (see [1, 7, 8, 13, 16, 19, 21–23]). The variable order calculus is a natural extension of the constant order (integer or fractional) calculus [24, 25]. The concepts of fractional differentiation and integration of variable order fractional are suggested by Samko and Ross in [20]. There are several proposed definitions for the variable order fractional derivative definitions; for more details, see [5, 6, 9–11]. In general, the variable-order fractional derivative is an extension of

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constant-order fractional derivative, where maybe the order function depends on one parameter, such as space or time, or system of other parameters [20, 24]. The variable order differentials have been studied in several applications such as control of a nonlinear viscoelasticity oscillator and the motion of particles suspended in a viscous fluid with drag force determined; for more details, see [17, 18] and the references cited therein.

The main goal of this work is to solve variable order cable equation numerically by an accurate numerical method based on Crank-Nicolson method. In what follows, we give the definition of Riemann-Liouville variable order fractional derivatives and Grünwald-Letnikov variable order fractional derivatives, which will be used in our study.

Definition 1.1 ([17]) The Riemann-Liouville variable order fractional derivative is defined as

$$ {}_0\mathcal{D}_x^{\alpha(x)} f(x,t) = \frac{1}{\Gamma(n-\alpha(x))} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{f(x,\tau)}{(x-\tau)^{\alpha(x)-n+1}} \mathrm{d}\tau, \quad x>0, \tag{1.1} $$

where $n-1<\alpha(x)<n$.

Definition 1.2 ([14]) The Grünwald-Letnikov variable order fractional derivative is defined as

$$ {}_0\mathcal{D}_x^{\alpha(x)} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha(x)}} \sum_{k=0}^{[x/h]} \omega_k^{(\alpha(x))} f(x-hk), \quad x \ge 0, \tag{1.2} $$

where $[x/h]$ means the integer part of x/h and $\omega_k^{(\alpha(x))}$ are the normalized Grünwald weights which are defined by $\omega_k^{(\alpha(x))} = (-1)^k \binom{\alpha(x)}{k}$.

The layout of this article is as follows: In Section 2, the variable order fractional cable equation is discretizated based on Crank-Nicolson method and shifted Grünwald formula; In Section 3, the stability analysis of the proposed method is discussed; In Section 4, linear and nonlinear numerical examples of variable order fractional cable problem are considered to show the accuracy of the scheme. Conclusions are given in Section 5.

2 Crank-Nicolson Scheme for Solving Variable Order Fractional Cable Equation

In this section, we combine two schemes Crank-Nicolson method and shifted Grünwald formula to solve the variable order fractional cable equation [2] on the form:

$$ \frac{\partial u(x,t)}{\partial t} = {}_0\mathcal{D}_t^{1-\beta(x,t)} \frac{\partial^2 u(x,t)}{\partial x^2} - \mu \, {}_0\mathcal{D}_t^{1-\alpha(x,t)} u(x,t) + f(x,t), \tag{2.1} $$

subject to

$$ u(x,0) = g(x), \quad 0 \le x \le L, \tag{2.2} $$

$$ u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad 0 \le t \le t_{\max} \tag{2.3} $$

where $0<\beta(x,t)$, $\alpha(x,t)<1$, $\mu>0$ is a constant and ${}_0\mathcal{D}_t^{1-\gamma(x,t)}$ is the variable order fractional derivative defined by the Riemann-Liouville operator of order $1-\gamma(x,t)$, where $\gamma(x,t)$ is equal to $\beta(x,t)$ or $\alpha(x,t)$.

Before constructing the proposed scheme, let us consider the uniform grid points of the space and time are defined as follows:

$$ x_n = nh, \quad n = 0,1,\cdots,N; \quad t_m = m\tau, \quad m = 0,1,\cdots,M, $$