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THE GLOBAL ATTRACTOR FOR A VISCOUS WEAKLY DISSIPATIVE GENERALIZED TWO-COMPONENT μ-HUNTER-SAXTON SYSTEM*

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Abstract This article is concerned with the existence of global attractor of a weakly dissipative generalized two-component μ-Hunter-Saxton (gμHS2) system with viscous terms. Under the period boundary conditions and with the help of the Galerkin procedure and compactness method, we first investigate the existence of global solution for the viscous weakly dissipative (gμHS2) system. On the basis of some uniformly prior estimates of the solution to the viscous weakly dissipative (gμHS2) system, we show that the semi-group of the solution operator $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set. Moreover, we prove that the dynamical system $\{S(t)\}_{t\geq 0}$ possesses a global attractor in the Sobolev space $H^2(\mathbb{S})\times H^2(\mathbb{S})$.

Key words Generalized two-component μ-Hunter-Saxton system; viscous weakly dissipative; existence; global attractor; period boundary conditions

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1 Introduction

The μ-Hunter-Saxton (μHS) equation

$$\mu(u)_t - u_{xxt} + 2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx} = 0 \tag{1.1}$$

was first introduced by Khesin et al [34] to model the evolution of rotators in liquid crystals with self-interaction and external magnetic field, and the unknown function $u(x,t)$ is a time-dependent function on the circle $\mathbb{S}=\mathbb{R}/\mathbb{Z}$ and $\mu(u)=\int_{\mathbb{S}} u\mathrm{d}x$ denotes its mean. With the interactions of rotators and external magnetic field, it is shown [34] that the (μHS) equation is a generalization of the rotator equation. In [34], the (μHS) equation is also constructed to describe the geodesic flow on $\mathscr{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the following inner product

$$\langle u_1,u_2\rangle = \mu(u_1)\mu(u_2) + \int_{\mathbb{S}} u_{1,x}u_{2,x}\mathrm{d}x.$$

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Moreover, in [34, 38], it is proved that the (μHS) equation is bi-Hamiltonian and also admits both periodic one-peakon solution and the multi-peakons.The (μHS) equation (1.1), called μ-Camassa-Holm equation [25], is closely related to the following celebrated Camassa-Holm (CH) shallow water wave equation,

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{1.2}$$

The (CH) equation was originally introduced to model the unidirectional propagation of shallow water waves over a flat bottom [6, 32]. Moreover, the (CH) equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [11, 16]. It is shown that the (CH) equation is completely integrable [3, 5, 6, 12–14] and also has a bi-Hamiltonian structure [19, 39]. The (CH) equation gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group [33]. Its solitary waves are peaked [7]. The peaked solitons are orbital stable [10], and the explicit interaction of the peaked solitons is given in [4]. Moreover, the (μHS) equation can be viewed as a generalized equation lying mid-way between the (CH) equation and the Hunter-Saxton (HS) equation given by

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0. \tag{1.3}$$

The (HS) equation was first introduced for rotators in liquid crystals [28], which describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. In [48], it is shown that the orientation of the molecules can also be characterized by the vectors field $(\cos u(x,t), \sin u(x,t))$. The (HS) equation can be regarded as the limit of the (CH) equation arising in a different physical context [29]. Similar to the (CH) equation, the (HS) equation is also completely integrable [29] and has a bi-Hamiltonian structure [32, 40]. For the other article related to the (HS) equation, see for example [28, 39, 48] and the references therein.

Recently, parallel to the (CH) equation and the (HS) equation mentioned above, the (μHS) equation (1.1) has been extended to the following generalized two-component μ-Hunter-Saxton (gμHS2) system (see for example [21])

$$\begin{cases} y_t + \sigma(2yu_x + uy_x) + \gamma u_{xxx} + \rho\rho_x - Du_x = 0, \\ \rho_t + (\rho u)_x = 0, \\ y = \mu(u) - u_{xx}, \ \mu(u) = \displaystyle\int_{\mathbb{S}} u(x,t)\mathrm{d}x. \end{cases} \tag{1.4}$$

The unknown functions $u(x,t)$ and $\rho(x,t)$ are time-dependent periodic functions on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$; and $\mu(u)$ is the mean of u on $\mathbb{S}$, $\sigma \in \mathbb{R}$ is the new free parameter, and $D \geq 0$. By utilizing the tri-Hamiltonian duality approach [21, 40], the (gμHS2) system can be derived from the bi-Hamiltonian structures of the Ito equation [30]. It is known that the (gμHS2) system has two Hamiltonians which are provided by

$$\begin{cases} H_1 = \dfrac{1}{2}\displaystyle\int_{\mathbb{S}} (\rho^2 + yu)\mathrm{d}x, \\ H_2 = \dfrac{1}{2}\displaystyle\int_{\mathbb{S}} (\sigma u u_x^2 + u\rho^2 - Du^2 + 2\sigma\mu(u)u^2 + \gamma u_x^2)\mathrm{d}x. \end{cases}$$

It is obvious that, by taking $\sigma = 1$, system (1.4) is reduced to the two-component μ-Hunter-Saxton (μHS2) system [49], which can be regarded as a μ-version of the two-component Dullin-Gottwald-Holm system [26]. It is shown in [49] that the (μHS2) system is a bi-variational