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LIOUVILLE THEOREM FOR CHOQUARD EQUATION WITH FINITE MORSE INDICES*

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Abstract In this article, we study the nonexistence of solution with finite Morse index for the following Choquard type equation

$$-\Delta u=\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\alpha}\,\mathrm{d}y|u(x)|^{p-2}u(x)\quad\text{in}\quad\mathbb{R}^N,$$

where $N\geq 3$, $0<\alpha<\min\{4,N\}$. Suppose that $2<p<\frac{2N-\alpha}{N-2}$, we will show that this problem does not possess nontrivial solution with finite Morse index. While for $p=\frac{2N-\alpha}{N-2}$, if $i(u)<\infty$, then we have $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y<\infty$ and $\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y$.

Key words Liouville type theorem; Morse index; Choquard equation

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1 Introduction

In the studying of the existence results for non-variational elliptic equations, we usually use the topological method such as the Leray-Schauder degree theory to get the existence result. In order to apply such a theory, the most important step is to get a priori bound for the solutions. As far as we know, the blow up method is the most powerful tool in proving priori bounds for elliptic equations. The idea of the blow up method is very simple and its essence is the proof by contradiction. More precisely, suppose on the contrary that there exists a sequence of solutions $\{u_n\}$ with $M_n=u_n(x_n)=\|u_n\|_{L^\infty(\Omega)}\to\infty$, then we make a scaling on this sequence of solutions and get $v_n(x)=\frac{1}{M_n}u_n(M_n^kx+x_n)$. Clearly, after the scaling, $\{v_n\}$ is bounded in the L^∞ norm. Hence, by the regularity theory of elliptic equations, we can assume that $v_n\to v$ in $C^{2,\gamma}_{loc}(\Omega_\infty)$ for some $0<\gamma<1$. Moreover, we have $\|v\|_{L^\infty}=1$ and it satisfies some limit equation in Ω_∞, where either $\Omega_\infty=\mathbb{R}^N$ or $\Omega_\infty=\mathbb{R}^N_+$ depending on the speed of x_n goes to the boundary of Ω. On the other hand, if we can prove the limit equations do not possess nontrivial solution, then we get a contradiction, hence the solutions of the original problem must be bounded. From the descriptions of the blow-up procedure, it is easy to see that in order to get a contradiction, it is essential to prove the Liouville type theorems for the limit equations.

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For the above mentioned reasons, Liouville theorems for elliptic equations have drawn much attention of scientists during the past few decades and there are many results on this subject up to now. For example, in two seminal articles [13, 14], Gidas and Spruck studied the nonexistence of positive solution for the following nonlinear elliptic problem

$$-\Delta u=|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N. \tag{1.1}$$

They proved that for the subcritical case, that is, $1<p<\frac{N+2}{N-2}$, this problem possesses no positive solution. This is the so-called Liouville type theorem for positive solution of problem (1.1). Later, in order to get the priori bound for elliptic equations in bounded domains, they studied a similar equation in the half space,

$$\begin{cases} -\Delta u=|u|^{p-1}u & \text{in} \quad \mathbb{R}^N_+, \\ u=0 & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases} \tag{1.2}$$

Similar nonexistence result was established in [14] for positive solution of the subcritical problem in the half space. The proof of Gidas and Spruck is very complicated. Later, W.Chen and C.Li [4] simplified their proofs and got similar results by using the moving plane method. After their results, the moving plane method and its variant, the moving sphere method were widely used in proving the Liouville theorems for elliptic equations; we refer the readers to [2, 3, 5–8, 11, 19, 20, 23] and we can not list all of them.

We note that all the results mentioned above only claim that the subcritical problems do not possess positive solution. A natural and more complicated question is that whether these problems possess sign-changing solutions. However, this problem is completely open up to now. The main difficulty lies in that the moving plane method does not work for sign-changing solutions. Hence, we must turn to other methods. A great progress on this area is the work [1], in which the authors studied the nonexistence of solution with finite Morse index for problems (1.1) and (1.2). They proved problems (1.1) and (1.2) do not possess nontrivial bounded solution with finite Morse index provided $1<p<\frac{N+2}{N-2}$. This result extended the nonexistence results of positive solution to finite Morse index solution. After the work [1], there are plenty of works concerning the finite Morse index solutions for elliptic equations. For example, A.Harrabi, S.Rebhi, and S.Selmi extended their results to more general nonlinear problems in [17, 18]. Recently, A.Harrabi, M.Ahmedou, S.Rebhi, and A.Selmi studied the nonexistence result for Neumann boundary value problems in [15]; X.Yu studied the mixed boundary problems in [24], the nonlinear boundary value problem in [25], and fractional Laplacian equation in [26]. X.Zhao and X.Wang obtained the nonexistence result for Robin boundary value problems in [27]. Other results can be found in [10, 12, 16] and the references therein.

Recently, we studied the nonexistence of positive solution for the following nonlocal equation

$$-\Delta u=\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\alpha}\,\mathrm{d}y|u(x)|^{p-2}u(x) \quad \text{in} \quad \mathbb{R}^N \tag{1.3}$$

in [28]. This kind of equation is usually called the Choquard type equation since in 1976, a similar equation as (1.3) was used by P.Choquard to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [21]. In some contexts, equation of type (1.3) is also called the nonlinear Schrödinger-Newton equation. In [28], we proved this equation does not possess positive solution for $0<p<\frac{2N-\alpha}{N-2}$ by using