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SMALL DATA SOLUTIONS OF 2-D QUASILINEAR WAVE EQUATIONS UNDER NULL CONDITIONS*

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Abstract For the 2-D quasilinear wave equation $(\partial_t^2-\Delta_x)u+\sum\limits_{i,j=0}^{2} g^{ij}(\partial u)\partial_{ij}u=0$ satisfying null condition or both null conditions, a blowup or global existence result has been shown by Alinhac. In this paper, we consider a more general 2-D quasilinear wave equation $(\partial_t^2-\Delta_x)u+\sum\limits_{i,j=0}^{2} g^{ij}(u,\partial u)\partial_{ij}u=0$ satisfying null conditions with small initial data and the coefficients depending simultaneously on u and ∂u. Through construction of an approximate solution, combined with weighted energy integral method, a quasi-global or global existence solution are established by continuous induction.

Key words global existence; null condition; weighted energy estimate; continuous induction

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1 Introduction and Main Results

We consider this second order quasilinear wave equation in $(0,\infty)\times\mathbb{R}^2$,

$$\begin{cases}(\partial_t^2-\Delta_x)u+\sum\limits_{i,j=0}^{2} g^{ij}(u,\partial u)\partial_{ij}u=0,\\ u(0,x)=\varepsilon u_0(x)+O(\varepsilon^2),\\ \partial_t u(0,x)=\varepsilon u_1(x)+O(\varepsilon^2),\end{cases} \tag{1.1}$$

where $x_0=t$, $x=(x_1,x_2)$, $\partial=(\partial_0,\partial_1,\partial_2)$, $\varepsilon>0$ is a sufficiently small constant, $u_0(x),u_1(x)$ are smooth functions supported in $|x|\le M$ and $(u_0(x),u_1(x))\not\equiv 0$, $g^{ij}(u,\partial u)=g^{ji}(u,\partial u)$ are

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still smooth functions and supposed to have the form

$$g^{ij}(u,\partial u)=\sum_{k=0}^{2}d_k^{ij}\partial_k u+\sum_{k=0}^{2}\mathrm{e}_k^{ij}u\partial_k u+\sum_{k,l=0}^{2}\mathrm{e}_{kl}^{ij}\partial_k u\partial_l u+O(|\partial u|^3), \tag{1.2}$$

with d_k^{ij}, e_k^{ij} and $\mathrm{e}_{kl}^{ij}=\mathrm{e}_{lk}^{ij}$ being some constants. Here and below $\partial_k u\partial_l u$ means $(\partial_k u)\partial_l u$.

Now we recall null conditions in two space dimensions defined in [1].

Definition 1.1 We say that $\sum\limits_{i,j,k=0}^{2}g_k^{ij}\partial_k u\partial_{ij}v$ and $\sum\limits_{i,j,k,l=0}^{2}h_{kl}^{ij}\partial_k u\partial_l v\partial_{ij}w$ satisfy the first null condition and the second null condition, respectively, if

$$\sum_{i,j,k=0}^{2}g_k^{ij}\omega_k\omega_i\omega_j\equiv 0\quad\text{and}\quad\sum_{i,j,k,l=0}^{2}h_{kl}^{ij}\omega_k\omega_l\omega_i\omega_j\equiv 0$$

hold for $(\omega_0,\omega_1,\omega_2)=(-1,\cos\theta,\sin\theta)$ with $0\le\theta\le 2\pi$.

For convenience, let

$$d(\omega)=\sum_{i,j,k=0}^{2}d_k^{ij}\omega_k\omega_i\omega_j,\quad e(\omega)=\sum_{i,j,k=0}^{2}\mathrm{e}_k^{ij}\omega_k\omega_i\omega_j\quad\text{and}\quad f(\omega)=\sum_{i,j,k,l=0}^{2}\mathrm{e}_{kl}^{ij}\omega_k\omega_l\omega_i\omega_j,$$

where $\omega_0=-1$ and $\omega=(\omega_1,\omega_2)\in\mathbb{S}^1$.

So far, the following results have been established.

- If all $d_k^{ij}=\mathrm{e}_k^{ij}=0$ in (1.2), then (1.1) has a global smooth solution if the null condition $f(\omega)\equiv 0$ holds, otherwise, the smooth solution of (1.1) blows up in finite time. See [2, 3].

- If all $d_k^{ij}=0$, but $\mathrm{e}_k^{ij}\neq 0$ for some (i,j,k) and $\mathrm{e}_{kl}^{ij}\neq 0$ for some (i,j,k,l) in (1.2), a global existence has been established in [3] under null conditions $e(\omega)=f(\omega)\equiv 0$ and a blow up result was proved in [4, 5], both assuming the term $\sum\limits_{i,j=0}^{2}g^{ij}\partial_{ij}u$ is independent of $\partial_t^2 u$.

- If all $\mathrm{e}_k^{ij}=0$, but $d_k^{ij}\neq 0$ for some (i,j,k) and $\mathrm{e}_{kl}^{ij}\neq 0$ for some (i,j,k,l) in (1.2), the following blowup and global results have been established.

i) Assume the null condition $d(\omega)\equiv 0$ holds. When $f(\omega)\not\equiv 0$, there exists a positive constant τ_0 depending only on $(u_0(x),u_1(x))$ and the coefficients $d_k^{ij},\mathrm{e}_{kl}^{ij}$ in (1.2) such that $\lim\limits_{\varepsilon\to 0^+}\varepsilon^2\ln T_\varepsilon=\tau_0$. Here T_ε denotes the lifespan of the smooth solution u to (1.1). One can refer to [6–10] and so on.

ii) Assume both null conditions $d(\omega)=f(\omega)\equiv 0$ hold. Then the smooth solution of (1.1) exists globally. See [1].

In this paper, we mainly consider the case $d_k^{ij}\neq 0$ for some (i,j,k), $\mathrm{e}_k^{ij}\neq 0$ for some (i,j,k) and $\mathrm{e}_{kl}^{ij}\neq 0$ for some (i,j,k,l) in (1.2) and address this problem: suppose null conditions $d(\omega)\equiv 0$ and $e(\omega)\equiv 0$ hold, does the smooth solution of (1.1) blow up in finite time when $f(\omega)\not\equiv 0$ or exist globally when $f(\omega)\equiv 0$?

Remark 1.2 For the case all $d_k^{ij}=0$, but $\mathrm{e}_k^{ij}\neq 0$ for some (i,j,k) and $\mathrm{e}_{kl}^{ij}\neq 0$ for some (i,j,k,l) in (1.2), the proofs of Theorems 1.3 and 1.4 below are almost the same as the case $d_k^{ij}\neq 0$ for some (i,j,k), $\mathrm{e}_k^{ij}\neq 0$ for some (i,j,k) and $\mathrm{e}_{kl}^{ij}\neq 0$ for some (i,j,k,l) in (1.2). We also give such comments in Remark 4.3 and proof of Theorem 1.3 in Section 5 and proof of Theorem 1.4 in Section 6.

Throughout this paper, we now assume the wave equation (1.1) satisfies null conditions $d(\omega)\equiv 0$ and $e(\omega)\equiv 0$. The main results in this paper are as follow.