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EULER SCHEME AND MEASURABLE FLOWS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS*

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Abstract For a stochastic differential equation with non-Lipschitz coefficients, we construct, by Euler scheme, a measurable flow of the solution, and we prove the solution is a Markov process.

Key words stochastic differential equation; Euler scheme; measurable flow; Markov property; non-Lipschitz

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1 Introduction

The object of the the present paper is the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X(t) = \sigma(t, X(t))\mathrm{d}w(t) + b(t, X(t))\mathrm{d}t, \quad t \in \mathbb{R}_+, \\ X(0) = x, \end{cases} \tag{1.1}$$

where $\{w_t, t \geqslant 0\}$ is a standard $\{\mathcal{F}_t\}$-d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual condition, σ and b are mappings from $\Omega \times [0, \infty) \times \mathbb{R}^n$ to $\mathbb{R}^n \otimes \mathbb{R}^d$ and $\mathbb{R}^n$ respectively. Precise conditions on σ and b will be given in the next section.

If the coefficients σ and b are Lipschitz in x, it is classic that the equation admits a unique solution. Moreover, this solution can be constructed by Picard iteration or Euler scheme, which is important in theoretical analysis as well as in numerical computation of the solution.

In [7] Watanabe and Yamada weakened the Lipschitz condition and proved that the existence of a unique solution under some non-Lipschitz condition. Yet, their proof is not constructive in the sense that the existence of the unique strong solution follows from the existence of a weak solution plus the pathwise uniqueness. Then, some forty years later, Zhang in [8] and Liu in [4] proved the convergence of the Euler scheme for the equation under conditions slightly stronger than those of [7] but weaker than the Lipschitz condition. In this respect let us mention also the recent paper [3] which claims the convergence of Euler scheme for equations under non-Lipschitz conditions comparable to those in [7]. But there is a gap in the proof

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which appears on page 4040 where the authors used the "fact" that for a sequence of random variables, say, ξ_n, there exists a subsequence $\{n_k\}$ such that

$$\limsup_{n\to\infty} \xi_n = \lim_{k\to\infty} \xi_{n_k},$$

which is, of course, false. Since this is a crucial step in their reasoning, the main result in that paper is, unfortunately, not proved.

The first aim of the present paper is to prove the convergence of the Euler scheme and the limit is the unique solution of (1.1) under assumptions similar to, but not completely the same, as those in [7]. Compared with [4, 8], main features of our results are that we allow more general non-lipschitz continuity modulus in x of the coefficients which may also depend on t possibly only in a measurable way. However, the convergence will be in probability, but not almost sure as in [4, 8]. So we assume less and obtain less. But, this is still enough for constructing the unique strong solution, which, to the best of our knowledge, has not been done ever before.

The second aim is the flow property of the solution. In the Lispchitz case, it is classic that the solution constitutes a stochastic flow of homeomorphisms and many useful properties, such as the Markov one, follow easily from this flow property. While in the non-Lipschitz case, one cannot expect that such a nice modification exists and thus it has not been clear whether or not the flow property holds true. Nevertheless, we will prove that there exists a version of the solution which preserves the flow property. Perhaps we may call it a hemi-flow since it is only jointly measurable in (t, x, ω) but loses the almost sure continuity in x (if it is continuous almost surely in x but not a homeomorphism, then it is called a semi-flow in [6]). This is unhappy, yet this hemi-flow may still be expected to play useful roles in some problems. In particular, as we will show, it yields the Markov property of the solution.

To conclude the introduction, let us mention that for stochastic differential equations with smooth coefficients, quasi-sure convergence of the Euler scheme is proved in [2].

2 Results

Keeping the notations in the introduction, we assume that

(A1) for all $t \in \mathbb{R}_+$, $\sigma(t,\cdot)$ and $b(t,\cdot)$ are continuous in x and there exists a positive function $\alpha(t)$ and an increasing concave function $\varphi : (0,\infty) \mapsto (0,\infty)$ with

$$\int_0^t \alpha(s)\mathrm{d}s < \infty,$$

$$\int_1^\infty \frac{\mathrm{d}x}{\varphi(x)} = \infty,$$

such that

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 \leqslant \alpha(t)(1+\varphi(|x|^2));$$

(A2) there exists an increasing concave function $\psi : [0,\infty) \mapsto [0,\infty)$ satisfying

$$\int_{0+} \frac{1}{\psi(u)}\mathrm{d}u = \infty,$$

such that for all $t > 0$,

$$|b(t,x) - b(t,y)|^2 + \|\sigma(t,x) - \sigma(t,y)\|^2 \leqslant \beta_r(t)\psi(|x-y|^2),$$