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CONTINUOUSLY DECREASING SOLUTIONS FOR A GENERAL ITERATIVE EQUATION*

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Abstract Most known results on polynomial-like iterative equations are concentrated to increasing solutions. Without the uniformity of orientation and monotonicity, it becomes much more difficult for decreasing cases. In this paper, we prove the existence of decreasing solutions for a general iterative equation, which was proposed as an open problem in [J. Zhang, L. Yang, W. Zhang, Some advances on functional equations, Adv. Math. (China) 24 (1995) 385-405] (or [W. Zhang, J.A. Baker, Continuous solutions of a polynomial-like iterative equation with variable coefficients, Ann. Polon. Math. 73 (2000) 29-36]).

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1 Introduction

Let X be a topological space. For any integer $n \geq 0$ the nth iterate of a self-mapping $f : X \to X$ is defined by $f^n = f \circ f^{n-1}$ and $f^0 = \mathrm{id}|_X$, where $\mathrm{id}|_X$ denotes the identity mapping. As shown in [2, 21], the linear combination of iterates of unknown function

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x) \quad \text{for} \quad x \in X \tag{1.1}$$

is called the polynomial-like iterative equation, where $F : X \to X$ is a given mapping and all coefficients λ_i $(i = 1, \cdots, n)$ are real constants. Equation (1.1) is discussed extensively, and there are lots of results on the existence, uniqueness and stability of its solutions (see [1, 3–6, 10–13, 15, 16, 19, 20, 22, 23, 25]), as well as multivalued and higher dimensional results (see [7, 9, 11, 14, 17, 18, 26]).

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Concerning equation (1.1), most known work are concentrated to increasing F and f. As mentioned in [6, 21, 24], a decreasing function reverses the orientation, and then the monotonicity is lost under iteration, which makes much difficulties in solving equation (1.1). This question was first answered in [20], by fixed point principle the authors proved the existence of decreasing solutions for equation (1.1) when F is decreasing. Recently, for the case that F is increasing, decreasing solutions with even terms were found by construction, where the hypotheses of local linearity on F was posed (see [10]).

In this paper, we continue to investigate decreasing solutions. More concretely, we discuss the following general iterative equation

$$G(f(x), f^2(x), \cdots, f^n(x)) = F(x), \tag{1.2}$$

which includes equation (1.1) as its special case. Without the assumption of locally linearity on the known function F, we give a general construction of decreasing C^0 solutions for equation (1.2) via Schröder transformation. Moreover, equation (1.2) is considered for all integer $n > 1$, which generalizes the existence result of decreasing solutions in [10]. Our paper is organized as follows. We first present some basic definitions and results in Section 2. Section 3 is contributed to the main results for F is increasing and decreasing, respectively. Then, based on our theorems, two examples will be given in Section 4.

2 Preliminaries

Let $x_1, x_2, \cdots, x_n \in \mathbb{R}$, denote $\vec{x} := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Especially denote $\vec{O} := (0, 0, \cdots, 0) \in \mathbb{R}^n$.

For $a \neq b \in \mathbb{R}$ and $n \in \mathbb{N}$, denote $\overrightarrow{(a, b)_n} := (a, b, a, b, \cdots) \in \mathbb{R}^n$, it is easy to see that a is located in odd position and b is located in even position. For instance, $\overrightarrow{(1, 2)_4} = (1, 2, 1, 2)$ and $\overrightarrow{(1, 2)_5} = (1, 2, 1, 2, 1)$.

Let $I_0 := [-1, 1]$ and J be a subset of $\mathbb{R}$. Let $C^1(I_0^m, J)$ denote the set of all C^1 maps from I_0^m to J, where $m \in \mathbb{N}$. Clearly, $C^1(I_0, \mathbb{R})$ is a Banach space with $\|h\|_1 = \max\limits_{t \in I_0}\{\|h\|, \|h'\|\}$, $h \in C^1(I_0, \mathbb{R})$, where $\|h\| = \sup\limits_{t \in I_0}\{h(t)\}$ and $\|h'\| = \sup\limits_{t \in I_0}\{h'(t)\}$. For $G \in C^1(I_0^m, \mathbb{R})$, denote

$$G'_i(\vec{x}) := \frac{\partial G}{\partial x_i}(\vec{x}), \quad i = 1, 2, \cdots, m.$$

In particular, we use the notation $\mathcal{H}_+(I_0, J)$ for the class of continuously differentiable functions $F : I_0 \to J$, which are strictly increasing, and $\mathcal{H}_-(I_0, J)$ denotes the strictly decreasing class.

Since we want to discuss the decreasing solutions of equation (1.2) we need to put some restrictions on these solutions. The case that f is a constant won't be considered. Obviously a decreasing solution has a unique fixed point, without any loss of generality, we assume the fixed point is origin and the domain of f is I_0. Moreover by suitable transformations we assume $f(-1) = 1$ and $f(1) = -1$.

If f is a decreasing solution of equation (1.2) then $G(\vec{O}) = F(0)$. We can assume that $G(\vec{O}) = 0$ and $F(0) = 0$. Otherwise if $G(\vec{O}) = F(0) = b \neq 0$, we can take $\widetilde{G} = G - b$ and $\widetilde{F} = F - b$ instead of G and F.