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SOLUTIONS TO QUASILINEAR HYPERBOLIC CONSERVATION LAWS WITH INITIAL DISCONTINUITIES*

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Abstract We study the singular structure of a family of two dimensional non-self-similar global solutions and their interactions for quasilinear hyperbolic conservation laws. For the case when the initial discontinuity happens only on two disjoint unit circles and the initial data are two different constant states, global solutions are constructed and some new phenomena are discovered. In the analysis, we first construct the solution for $0 \le t < T^*$.Then, when $T^* \le t < T'$, we get a new shock wave between two rarefactions, and then, when $t > T'$, another shock wave between two shock waves occurs. Finally, we give the large time behavior of the solution when $t \to \infty$. The technique does not involve dimensional reduction or coordinate transformation.

Key words singular structure; quasilinear hyperbolic equations; elementary wave; global solutions

2010 MR Subject Classification 35L65; 35L67

1 Introduction

Riemann solution is one of the building blocks for solutions to hyperbolic conservation laws. The method of constructing self-similar solutions is very powerful in the study of one dimensional Riemann problem. In this paper, we will construct solutions to multi-dimensional hyperbolic conservation laws without the self-similar structure.

The equation of n-dimensional conservation law has the following form

$$u_t + \sum_{i=1}^{n} \frac{\partial f_i(u)}{\partial x_i} = 0, \tag{1.1}$$

with the initial value

$$u(x,t)|_{t=0} = u_0(x) = \begin{cases} u_-, \, M(x) < 0, \\ u_+, \, M(x) > 0, \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, \ldots, x_n), u = u(x,t), f_i(u) \in C^2(R), i = 1, 2, \ldots, n, M(x) \in C^1(R^n)$, and $M(x) = 0$ is a smooth surface dividing R^n into two infinite parts.

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The Cauchy problem for equation (1.1) was studied by Conway and Smoller [3]. Kruzkov [8], Vol'pert [5, 6, 9, 13, 15], and they proved weak solution uniquely exists, under some kind of entropy condition. Here, (1.1)–(1.2) can be considered as another generalization of one dimensional Riemann problem. In 1-D case, u_- and u_+ are disjoint at a point, while the initial discontinuity is usually a curve in 2-D case or a surface in 3-D case etc. In [16], the author constructed shock wave and rarefaction wave solutions, especially, the rarefaction wave is obtained by implicit function theorem, instead of the self-similar approach.

In this paper, we will study the singular structure of n-dimensional global solutions and wave interactions of n-dimensional elementary waves from a different perspective. More precisely, we assume that the initial discontinuity is a smooth n-dimensional surface and initial values are two different constant states. This is a basic problem for investigating structure and development of global solution. Obviously, this kind of solutions are in general not self-similar. For this, in [16], a new multi-dimensional method was proposed to study such kind of multi-dimensional non-selfsimilar solutions, and the main results of [16] can be summarized as follows.

Definition 1.1 (see [8]) $u(x,t)$ is a weak solution of (1.1) and (1.2), if

$$\int_{R^+}\int_{R^n}\left(u\frac{\partial\phi}{\partial t}+\sum_{i=1}^{n}f_i(u)\frac{\partial\phi}{\partial x_i}\right)\mathrm{d}x\mathrm{d}t+\int_{t=0}u_0(x)\phi(x,0)\mathrm{d}x=0, \tag{1.3}$$

for all test functions $\phi\in C_0^\infty(R^n\times R^+)$.

Definition 1.2 (see [16]) (Condition H (H$'$)) For all $x\in\{x|\,M(x)=0,x\in R^n\}$ and u between u_- and u_+, if the following form

$$\sum_{i=1}^{n}M_{x_i}f_i{}''(u)>0\ (<0) \tag{1.4}$$

holds, we call (1.1) and (1.2) satisfies condition H (H$'$), where, $M_{x_i}=M_{x_i}(x)\big|_{x\in\{x|\,M(x)=0\}}$, $i=1,2,\cdots,n$, $u\in(a,b)$, a,b can be finite or ∞.

Proposition 1.3 (see [16]) Assume condition H (H$'$) is satisfied, for $\forall x\in\{x|\,M(x)=0\}$.

If $u_->u_+$ $(u_-<u_+)$, u is a n-dimensional shock wave solution, given by

$$u(x,t)=\begin{cases}u_-, & M\left(x_1-\dfrac{[f_1]}{[u]}t,x_2-\dfrac{[f_2]}{[u]}t,\cdots,x_n-\dfrac{[f_n]}{[u]}t\right)<0,\\ u_+, & M\left(x_1-\dfrac{[f_1]}{[u]}t,x_2-\dfrac{[f_2]}{[u]}t,\cdots,x_n-\dfrac{[f_n]}{[u]}t\right)>0,\end{cases} \tag{1.5}$$

where $[u]=u_+-u_-,[f_i]=f_i(u_+)-f_i(u_-),i=1,2,\cdots,n$, the discontinuity is located at

$$M\left(x_1-\frac{[f_1]}{[u]}t,x_2-\frac{[f_2]}{[u]}t,\cdots,x_n-\frac{[f_n]}{[u]}t\right)=0. \tag{1.6}$$

Denote the shock wave as S. $M=0$ is the discontinuity of u_+ and u_-.

If $u_-<u_+(u_->u_+)$, u is a n-dimensional rarefaction wave with

$$u(x,t)=\begin{cases}u_-, & M(x_1-f_1'(u_-)t,x_2-f_2'(u_-)t,\cdots,x_n-f_n'(u_-)t)<0,\\ R(x,t), & M(x_1-f_1'(u_-)t,x_2-f_2'(u_-)t,\cdots,x_n-f_n'(u_-)t)\ge 0\\ & \text{and } M(x_1-f_1'(u_+)t,x_2-f_2'(u_+)t,\cdots,x_n-f_n'(u_+)t)\le 0,\\ u_+, & M(x_1-f_1'(u_+)t,x_2-f_2'(u_+)t,\cdots,x_n-f_n'(u_+)t)>0,\end{cases} \tag{1.7}$$