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GLOBAL STABILITY OF TRAVELING WAVEFRONTS FOR NONLOCAL REACTION-DIFFUSION EQUATIONS WITH TIME DELAY*

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Abstract This paper is concerned with the stability of traveling wavefronts for a population dynamics model with time delay. Combining the weighted energy method and the comparison principle, the global exponential stability of noncritical traveling wavefronts (waves with speeds $c > c_*$, where $c = c_*$ is the minimal speed) is established, when the initial perturbations around the wavefront decays to zero exponentially in space as $x \to -\infty$, but it can be allowed arbitrary large in other locations, which improves the results in [9, 18, 21].

Key words nonlocal reaction-diffusion equations; traveling wavefronts; stability; comparison principle; weighted energy method

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1 Introduction

In this paper, we consider the following nonlocal reaction-diffusion equation with time delay

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + f((g*u)(t,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

with the initial data

$$u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in \mathbb{R}, \tag{1.2}$$

where

$$f(w) = pw\mathrm{e}^{-aw},$$

and

$$(g*u)(t,x) = \int_{-\infty}^{+\infty} g(y)u(t-\tau, x-y)\mathrm{d}y.$$

This model represents the population distribution of single species such as the Australian blowfly [8, 10, 15], which is derived from the original delay ODE model [3] based on Nicholson's experimental data [13, 14]. Here, $u(t,x)$ denotes the mature population at time t and location x, $\delta > 0$ is the death rate of the mature population, $\tau > 0$ is the maturation time (the time required for a newborn to become matured). $f(w)$ is Nicholson's birth function, $a > 0$ is a

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constant, $p > 0$ is the impact of the death on the immature population. $g(x)$ is the heat kernel in the form of

$$g(x) = \frac{1}{\sqrt{4\pi\rho}} \mathrm{e}^{-\frac{x^2}{4\rho}} \quad \text{with} \quad \int_{-\infty}^{+\infty} g(x)\mathrm{d}x = 1.$$

When $g(x)$ is replaced by the Dirac-$\delta(x)$ function, (1.1) reduces to the local Nicholson's blowflies equation with discrete delay

$$u_t(t,x) = \Delta u(t,x) - \delta u(t,x) + pu(t-\tau,x)\mathrm{e}^{-au(t-\tau,x)}. \tag{1.3}$$

In recent years, traveling wave solutions of (1.1) and (1.3) were widely investigated (see [5–11, 16, 21]). A traveling wave solution of (1.1) and (1.3) is a special translation invariant solution of the form $u(t,x) = \phi(\xi)$, $\xi = x + ct$, where ϕ is the wave profile that propagates through the one-dimensional spatial domain at a constant velocity $c > 0$. If $\phi(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called a traveling wavefront. In the study of traveling wave solutions, the stability of traveling wave solutions is an important and difficult object. We refer the readers to [2, 4, 6, 7, 10–12, 17, 18, 20–24].

For the local equation (1.3), the stability of traveling wave solutions has been well done. We provide some background in two cases: $1 < p/\delta \le \mathrm{e}$ and $p/\delta > \mathrm{e}$. In the first case, the birth rate function $f(u) = pu\mathrm{e}^{-au}$ is monotonically increasing for $u \in [0, \frac{1}{a}\ln\frac{p}{\delta}]$. In 2004, Mei et al. [10] proved that the wavefronts of (1.3) are stable for the wave speed $c > 2\sqrt{p-\delta}$ with a sufficiently small initial perturbation by the weighted energy method. Later on, by the comparison principle together with the weighted energy method, Lin and Mei [6] improved the stability of traveling waves to $c > c_*$ (c_* is minimal speed), when the time delay $\tau \ll 1$, and the initial perturbation around the wavefront decays to zero exponentially in space as $x \to -\infty$, but it can be large in other locations. In [11], Mei et al. further obtained the stability for all waves of (1.3) with speed $c > c_*$ but no restriction is needed for the time delay τ. The approach they used is still the weighted energy method together with the comparison principle, but the weight function is different from that in [6]. Recently, by the weighted energy method and the Green function technique, Mei, Ou and Zhao [12] improved the stability results in [11], and proved that all noncritical wavefronts (waves with speed $c > c_*$) are globally exponentially stable, and critical wavefronts (waves with speed $c = c_*$) are globally algebraically stable when the initial perturbations around the wavefront decay to zero exponentially near the negative infinity regardless of the magnitude of time delay. In the second case, the birth rate $f(u)$ is nonmonotone on $[0, \frac{1}{a}\ln\frac{p}{\delta}]$. Lin et al. [7] used the technical weighted energy method to prove the exponential stability of all noncritical traveling waves. In [1], Chern et al. still applied the technical weighted-energy method, but with some new flavors to handle the critical oscillatory waves, and proved that the critical traveling waves (monotone or oscillatory) are also time-asymptotically stable.

For the nonlocal equation (1.1) with monotone birth function, the stability of traveling wavefronts was investigated in [9, 18, 20, 21]. Wu et al. [20] studied a nonlocal reaction-diffusion equation with delay

$$u_t = Du_{xx} - h(u(x,t)) + f\left(\int_{\mathbb{R}} g(y)S(u(x-y,t-\tau))\mathrm{d}y\right). \tag{1.4}$$

By the (technical) weighted energy method, they proved that the traveling wavefront of (1.4) is exponentially stable, when the initial perturbation around the wave is suitable small in a