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STOCHASTIC HEAT EQUATION WITH FRACTIONAL LAPLACIAN AND FRACTIONAL NOISE: EXISTENCE OF THE SOLUTION AND ANALYSIS OF ITS DENSITY*

Junfeng LIU (刘俊峰)
Department of Statistics, Nanjing Audit University, Nanjing 211815, China
E-mail: *junfengliu@nau.edu.cn*

Ciprian A. TUDOR
Laboratoire Paul Painlevé, Université de Lille 1, F-59655 Villeneuve d'Ascq, France
CIMFAV, Universidad de Valparaiso, Chile
E-mail: *tudor@math.univ-lille1.fr*

Abstract In this paper we study a fractional stochastic heat equation on $\mathbb{R}^d$ $(d \geq 1)$ with additive noise $\frac{\partial}{\partial t}u(t,x) = \mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}u(t,x) + b(u(t,x)) + \dot{W}^H(t,x)$ where $\mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}$ is a nonlocal fractional differential operator and $\dot{W}^H$ is a Gaussian-colored noise. We show the existence and the uniqueness of the mild solution for this equation. In addition, in the case of space dimension $d = 1$, we prove the existence of the density for this solution and we establish lower and upper Gaussian bounds for the density by Malliavin calculus.

Key words stochastic partial differential equation; fractional Brownian motion; Malliavin calculus; Gaussian density estimates

2010 MR Subject Classification 60G35; 60H07; 60H15

1 Introduction

In this article, we will focus on the following fractional stochastic heat equation with Cauchy initial condition on $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}u(t,x) + b(u(t,x)) + \dot{W}^H(t,x), \quad \text{in} \quad [0,T] \times \mathbb{R}^d, \\ \quad u(0,\cdot) = u_0(\cdot), \quad x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $d \geq 1$, $\underline{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is the order of the nonlocal fractional differential operator $\mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}$ and $\underline{\delta} = (\delta_1, \cdots, \delta_d)$ is its skewness parameter. The noise $\dot{W}^H(t,x)$ is the fractional-colored noise which behaves as a fractional Brownian motion in time and is correlated in space, characterized by its covariance function (2.2). The drift $b : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and the initial

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value $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a real-valued measurable continuous function. Throughout this paper, we will assume that α_i and δ_i are fixed constants such that for any $i = 1, \cdots, d$

$$\alpha_i \in]0, 2]/\{1\} \quad \text{and} \quad |\delta_i| \leq \min\{\alpha_i, 2 - \alpha_i\}. \tag{1.2}$$

In particular, in one space dimension, (i.e., $d = 1$), when $\delta = 0$, we know that (Debbi and Dozzi [11]) the operator $\mathcal{D}^{\alpha}_{0}$ coincides with the fractional Laplacian operator, which has also been widely studied. Various phenomena in physics (such as diffusions in a disordered or fractal medium), in image analysis, or in risk management have been modeled by means of stochastic (partial) differential equations with fractional Laplacian.

There exists an already vast literature on stochastic partial differential equations driven by (additive or multiplicative) Gaussian noises and involving various types of differential operators. The starting point of this literature is the stochastic heat equation (i.e., $\delta = 0$ and $\alpha_i = 2$ for $i = 1 \cdots d$) driven by time-space white noise (i.e., $H = \frac{1}{2}$ and f is the Dirac function in (2.2)). Then the scientific literature on the field has been developed in various directions: generalization of the noise in space ("colored noise", see [25] or [9]) or in time ("fractional noise", see among others, [1, 2, 4, 5, 12, 15–18]), or by considering differential operators more general than the Laplacian (see [10, 11]). We here generalize both the operator and the noise.

In general, the existence and the main properties of the solution depend on the smoothness of the noise W^H and on the properties of the differential operator $\mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}$. This is also the case in our paper. Our noise is fractional in time with Hurst parameter $H > \frac{1}{2}$ and "colored in space", i.e., has a correlated spatial structure that makes it smoother than the white noise. On the other hand, the operator $\mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}$ is rougher than the Laplacian. The main contribution of our work is to understand the combination of these two elements: the "smooth" Gaussian noise and the "rough" pseudo-differential operator. The first step is to prove the existence and uniqueness of the solution. We will show that the SPDE (1.1) admits a unique mild solution under the following.

Assumption 1 ($\mathbf{H}^{\alpha,2H}$)

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + \mathrm{Re}\Psi(\xi)} \right)^{2H} \mu(\mathrm{d}\xi) < +\infty, \tag{1.3}$$

where we denote by

$$\Psi(\xi) = \sum_{j=1}^{d} |\xi_j|^{\alpha_j} \exp\left(-\mathrm{i}\delta_j \frac{\pi}{2} \mathrm{sgn}(\xi_j) \right). \tag{1.4}$$

Here $\underline{\alpha}$ is the so-called stability parameter of the operator $\mathcal{D}^{\underline{\alpha}}_{\underline{\delta}}$, H is the Hurst parameter of the noise in time and μ is the measure that governs the spatial covariance of the noise. Condition (1.3) extends the findings in [1] for the case of the standard Laplacian. As shown in this reference, the condition (1.3) is sharp in the case of the linear heat equation with fractional-colored noise (see also [9] for the nonlinear heat equation with white noise in time), and thus we cannot expect to find a better condition in our case. Furthermore, we establish the lower and upper bounds for the density of the mild solution in the space dimension 1. To obtain the Gaussian type bounds for the density of the solution we will employ a useful density formula given in the paper in [19]. This has been used by other authors to obtain bounds for the density of the solution of a SPDE with additive Gaussian noise, see e.g., [17, 18, 22, 23] among others.