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DECAY RATE OF FOURIER TRANSFORMS OF SOME SELF-SIMILAR MEASURES*

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Abstract This paper is concerned with the Diophantine properties of the sequence $\{\xi\theta^n\}$, where $1 \le \xi < \theta$ and θ is a rational or an algebraic integer. We establish a combinatorial proposition which can be used to study such two cases in the same manner. It is shown that the decay rate of the Fourier transforms of self-similar measures μ_λ with $\lambda = \theta^{-1}$ as the uniform contractive ratio is logarithmic. This generalizes some results of Kershner and Bufetov-Solomyak, who consider the case of Bernoulli convolutions. As an application, we prove that μ_λ almost every x is normal to any base $b \ge 2$, which implies that there exist infinitely many absolute normal numbers on the corresponding self-similar set. This can be seen as a complementary result of the well-known Cassels-Schmidt theorem.

Key words self-similar measures; Fourier transforms; decay rate; normal numbers

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1 Introduction

In 1916, Weyl [1] proved that for any fixed $\theta > 1$, the sequence $\{\xi\theta^n\}$ is equidistributed or uniformly distributed modulo one for almost all real number ξ. Namely,

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{card}\{n : 1 \le n \le N, \xi\theta^n \pmod 1 \in I\} = |I|,$$

for any subinterval I of $[0, 1]$, where $|I|$ is the length of I. So how about the behavior of the specific sequence $\{(\frac{3}{2})^n\}$? Is it uniformly distributed modulo one? Indeed, it is not even known to be dense. Vijayaraghavan [2] in 1939 first showed that there are infinitely many limit points of the sequence $\{(\frac{p}{q})^n\}$, where p, q are relatively prime integers with $p > q \ge 2$. Almost at the same time, Pisot [3] obtained a similar result when θ is an algebraic number. The reader may refer to the references [4], [5] for more information. Based on their work, in 1992, Flatto observed that the above problem has some connections with the symbolic dynamics which is induced by the transformation $f : x \to \frac{p}{q}x \pmod 1, x \in [0, 1)$. Later, with the joint efforts of Lagarias and Pollington, Flatto showed that the gap between the largest and the smallest limit point is at least $\frac{1}{p}$ (see [6] or [7]). Similar results were obtained by Dubickas [8] when θ is an algebraic number.

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In this paper, we study the following problem. How many times do the values of the sequence $\{\xi\theta^n\}$ (for a fixed $\theta > 1$) fall into a prescribed region I among the time $1 \le n \le N$? This problem is tightly related to the Fourier transforms of some self-similar measures (with contractive ratio $\frac{1}{\theta}$), which we shall discuss later. We show that this problem is determined by the arithmetical property of θ in certain sense, and that the Diophantine properties of the sequence $\{\xi\theta^n\}$ behave in the similar manner when θ is a rational or an algebraic integer. We prove a combinatorial proposition which can be used to explain the common behavior, for the precise statement see Proposition 2.1. In the remainder of Section 2, we establish that both cases satisfy the hypothesis of Proposition 2.1, see Lemmas 2.2 and 2.4.

Bernoulli convolution ν_λ, as the distribution of the random sum $\sum \pm\lambda^n, 0 < \lambda < 1$, where the signs are chosen independently with probability $\frac{1}{2}$, have been studied for many years, see the survey [9]. It is natural to ask what is the relationship between the parameter λ and the Fourier transforms of the measure ν_λ (denoted by $\widehat{\nu}_\lambda$). It seems that this was first pointed out by Kershner, Erdős, and others in the late 1930s. They showed that the decay rate of $\widehat{\nu}_\lambda$ depends on the arithmetical property of λ. Specifically, Kershner [10] obtained the logarithmic decay rate of $\widehat{\nu}_\lambda$ when λ is a rational number but not of the form $\frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{q}, q \in \mathbb{N}$. And Erdős [11] proved that the decay rete of $\widehat{\nu}_\lambda$ is polynomial for a.e. λ sufficiently close to one. In this direction these are further investigated by many authors, such as Kahane [12] and others. Recently, Bufetov and Solomyak [13] also obtained that the decay rate of $\widehat{\nu}_\lambda$ is logarithmic for some classes of algebraic λ. We generalize and extend Kershner and Bufetov-Solomyak's results to self-similar cases (Bernoulli convolutions are special cases of self-similar measures). Before stating our results, we recall some definitions and notations.

Recall that an iterated function system (IFS) $\mathscr{I} = \{f_1, f_2, \cdots, f_m\}, m \ge 2$ is a finite family of strict contractive maps on $\mathbb{R}$ (actually for complete metric space, here we only consider the real line). Let $P = (p_1, p_2, \cdots, p_m)$ be a non-degenerate probability vector, i.e.,

$$\sum_{i=1}^{m} p_i = 1,\ 0 < p_i < 1 \text{ for all } 1 \le i \le m.$$

It is well known that there is a unique nonempty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{i=1}^{m} f_i(K)$$

and a unique Borel probability measure on K satisfying

$$\mu = \sum_{i=1}^{m} p_i f_i \mu,$$

where $f\mu := \mu \circ f^{-1}$ is the push-forward of μ by the transformation f. We say that K is the self-similar set and μ is the self-similar measure on it associated with the probability vector P. For general self-similar measures, the behaviours of Fourier transform are complicated. So we only concern the case when all contractive ratios are the same (homogeneous). More precisely, we assume that $f_i(x) = \lambda x + a_i,\ 0 < \lambda < 1$, all $a_i\ (1 \le i \le m),\ m \ge 2$ are distinct. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_m$. Denote μ_λ to be the self-similar measure associated with a non-degenerate probability vector P. Here and after, we study μ_λ with the above hypothesis.

Our main theorem is as following.