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ANOTHER CHARACTERIZATIONS OF MUCKENHOUPT A_p CLASS*

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Abstract This manuscript addresses Muckenhoupt A_p weight theory in connection to Morrey and BMO spaces. It is proved that ω belongs to Muckenhoupt A_p class, if and only if Hardy-Littlewood maximal function M is bounded from weighted Lebesgue spaces $L^p(\omega)$ to weighted Morrey spaces $M_q^p(\omega)$ for $1<q<p<\infty$. As a corollary, if M is (weak) bounded on $M_q^p(\omega)$, then $\omega\in A_p$. The A_p condition also characterizes the boundedness of the Riesz transform R_j and convolution operators T_ϵ on weighted Morrey spaces. Finally, we show that $\omega\in A_p$ if and only if $\omega\in \mathrm{BMO}^{p'}(\omega)$ for $1\le p<\infty$ and $1/p+1/p'=1$.

Key words characterization; Hardy-Littlewood maximal function; Muckenhoupt A_p class; weighted Morrey spaces; weighted BMO space

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1 Introduction

For $1<p<\infty$ and a nonnegative locally integrable function ω on $\mathbb{R}^n$, ω is in the Muckenhoupt A_p class if it satisfies the condition

$$[\omega]_{A_p}:=\sup_Q\left(\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{-\frac{1}{p-1}}\mathrm{d}x\right)^{p-1}<\infty.$$

A weight function ω belongs to the class A_1 if there exists $C>0$ such that for every cube Q,

$$\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\le C\operatorname*{ess\,inf}_{x\in Q}\omega(x),$$

and the infimum of C is denoted by $[\omega]_{A_1}$. A weight ω is called an A_∞ weight if

$$[\omega]_{A_\infty}:=\sup_Q\left(\frac{1}{|Q|}\int_Q\omega(x)\mathrm{d}x\right)\exp\left(\frac{1}{|Q|}\int_Q\log\omega(x)^{-1}\mathrm{d}x\right)<\infty.$$

In fact, $A_\infty=\bigcup\limits_{1\le p<\infty}A_p$.

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Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights plays an important role in the study of boundary value problems for Laplace's equation on Lipschitz domains. Other applications of weighted inequalities include vector-valued inequalities, extrapolation of operators and applications to certain classes of integral equation and nonlinear partial differential equation. There are a number of classical results demonstrating that the Muckenhoupt A_p classes are the right collections of weights to do harmonic analysis on weighted spaces. The main results along these lines are the equivalence between the $\omega \in A_p$ condition and the $L^p(\omega)$ boundedness (or weak boundedness) of maximal operator and singular integral operators.

A well known result of Muckenhoupt [16] showed that the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\mathrm{d}y$$

is (weak) bounded on weighted Lebesgue spaces $L^p(\omega)$ if and only if $\omega \in A_p$ for $1 < p < \infty$ (for the case $n = 1$). Hunt, Muckenhoupt and Wheeden [10] proved that the A_p condition also characterizes the $L^p(\omega)$ boundedness of the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\mathrm{d}y.$$

Later, Coifman and Fefferman [6] extended the A_p theory to the case $n \geq 1$ and general Calderón-Zygmund operators, they also proved that A_p weights satisfy the crucial reverse Hölder condition.

In 2009, Komori and Shirai [12] introduced the weighted Morrey spaces. Let $0 < q < p < \infty$, ω be a weight and $\omega(Q) := \int_Q \omega(x)\mathrm{d}x$. Then a weighted Morrey space is defined by

$$M_q^p(\omega) = \left\{ f \in L^q_{\mathrm{loc}}(\omega) : \|f\|_{M_q^p(\omega)} := \sup_Q \frac{1}{\omega(Q)^{1/q-1/p}}\left(\int_Q |f(x)|^q\omega(x)\mathrm{d}x\right)^{1/q} < \infty \right\},$$

and a weighted weak Morrey space is defined by

$$WM_q^p(\omega) = \left\{ f \in L^q_{\mathrm{loc}}(\omega) : \|f\|_{WM_q^p(\omega)} < \infty \right\},$$

where

$$\|f\|_{WM_q^p(\omega)} := \sup_Q \frac{1}{\omega(Q)^{1/q-1/p}} \sup_{\lambda>0} \lambda\Big(\int_{\{x\in Q:|f(x)|>\lambda\}} \omega(x)\mathrm{d}x\Big)^{1/q}.$$

They proved that if $\omega \in A_p$, then M is bounded on $M_q^p(\omega)$. An interesting question is raised. Is ω in A_p if M is bounded on $M_q^p(\omega)$ for $1 < q < p < \infty$? We will give an affirmative answer as follows.

Theorem 1.1 Let $1 < q < p < \infty$. The following statements are equivalent:

(1) $\omega \in A_p$;

(2) M is a bounded operator from $L^p(\omega)$ to $L^{p,\infty}(\omega)$;

(3) M is a bounded operator from $L^p(\omega)$ to $M_q^p(\omega)$;

(4) M is a bounded operator from $L^p(\omega)$ to $WM_q^p(\omega)$;

(5) M is a bounded operator from $M_q^p(\omega)$ to $M_q^p(\omega)$;

(6) M is a bounded operator from $M_q^p(\omega)$ to $WM_q^p(\omega)$.