Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2017,**37B**(6):1775–1790
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

LARGE TIME BEHAVIOR OF SOLUTIONS TO THE PERTURBED HASEGAWA-MIMA EQUATION*

Lijuan WANG (王利娟)
School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai 200260, China
E-mail: *ljwang66@163.com*

Abstract The large time behavior of solutions to the two-dimensional perturbed Hasegawa-Mima equation with large initial data is studied in this paper. Based on the time-frequency decomposition and the method of Green function, we not only obtain the optimal decay rate but also establish the pointwise estimate of global classical solutions.

Key words perturbed Hasegawa-Mima equation, Green function, time-frequency decomposition, pointwise estimates.

2010 MR Subject Classification 35A01; 35M11; 82D10

1 Introduction

In this paper we are concerned with the time-asymptotic behavior of solutions to the following two-dimensional (2D) perturbed Hasegawa-Mima equation,

$$\partial_t(u-\Delta u)+k\partial_{x_2}u-\lambda\Delta(u-\Delta u)=-\{u,\Delta u\},\quad (x_1,x_2)\in\mathbb{R}^2,\quad t>0, \tag{1.1}$$

where k and λ are constants, $0<\lambda<1$, Δ is the Laplacian operator, $u(x_1,x_2,0)=u_0(x_1,x_2)$ is the initial data and $\{\cdot,\cdot\}$ denotes the Poisson bracket

$$\{h,g\}=(\partial_{x_1}h)(\partial_{x_2}g)-(\partial_{x_2}h)(\partial_{x_1}g).$$

Equation (1.1) is the simplest and powerful 2D turbulent system. It not only describes the time evolution of drift wave in plasma but also describes the temporal evolution of geostrophic motion and is called the quasi-geostrophic potential vorticity equation for the Rossby wave in geophysical fluids. Moreover, if we eliminate the term u_t in (1.1), it will turn to Euler equation for the incompressible homogeneous fluids. The Euler equation was actively studied by many authors in recent years.

Recently, the perturbed Hasegawa-Mima equation receives much attention. Guo and Han [1] obtained global solutions to the Cauchy problem. Grauer obtained the energy estimate for the perturbed Hasegawa-Mima equation [2]. Zhang and her collaborator got the existence and the uniqueness of the global solution for the generalized Hasegawa-Mima equation in [3]. The author of this article also studied the pointwise estimates of solutions to (1.1) with small initial

*Received July 12, 2016; revised June 13, 2017. The research is supported by the National Natural Science Foundation of China (11231006).

data [4]. Guo and Han [1] obtained the global solutions to the Cauchy problem. In this paper, we considered the large time behavior of solutions to (1.1) in case of the large perturbation to the initial data. To the best of our knowledge, there has not been any work done on the asymptotic behavior in the case of large initial data. We attempt to explore methods of studying the large time behavior of solutions in the case of large perturbation in the present paper.

The main approaches in this paper are the time-frequency decomposition, Green function as well as the energy method. In fact, using Green function to study pointwise estimates for hyperbolic-parabolic systems became a very active field of research in recent years. Actually, pointwise estimates play a crucial role in the description of the evolution of the solution, as it gives the explicit expression of the time asymptotic behavior of solutions. Through the pointwise estimates, we can not only obtain the decay rate of the solution which is due to the parabolicity of the system, but also find out the movement of the main part of the solution which is caused by the hyperbolicity. The Green function method was first introduced by Liu and Zeng in [5] to get the pointwise estimates of solutions for one dimensional quasilinear hyperbolic-parabolic systems of conservation laws. Later, Hoff and Zumbru ([6] and [7]) studied the Navier-Stokes equation with viscosity. Liu and Wang [8] obtained the pointwise estimates of the solutions for the isentropic Navier-Stokes equations in odd dimensions. After that, a lot of papers studied the pointwise estimates of various types of hyperbolic-elliptic equations (see e.g. [9–14]). The classical Green function method is decomposing the Green function into three parts: lower frequency part, middle part and higher frequency part. Then by Taylor expansion, Fourier analysis and Kirchhoff formulas to obtain the pointwise estimate of each part. The pointwise estimate of the Green function shows that the large time behavior of Green function is dominated by lower frequency waves while the higher frequency wave play a much more significant role in short time.

We emphasize that the basic idea of using Green function method to study the Cauchy problem of nonlinear evolution equation with small perturbation is to use the smallness of the perturbation and iteration. The smallness of the perturbation guarantees the convergence of the iteration. Obviously, the idea does not work when the initial data is large. Without the smallness of the initial data $\|u_0\|_{L^2}$, the difficulty here is how to estimate the nonlinear term of equation (1.1). Fortunately, based on the special structure of the equation itself, we can easily get the boundedness of $\|u\|_{H^2}$. Then, by taking a time-frequency decomposition we get the decay rate of $\|u\|_{H^2}$. After that, by energy method and Green function method alternately, we obtain the decay rate of high-order derivatives of $u(x,t)$. Finally, taking advantage of decay rate of $\|D_x^\alpha u\|_{L^2}$ and Green function method, the optimal decay estimate of $\|D_x^\alpha u\|_{L^2}$ is established. In addition, with those optimal decay estimates in hand, we obtain pointwise estimates of solutions.

The main result in this paper is stated as follows.

Theorem 1.1 Suppose $u_0 \in H^s(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $s > 2$, and $|D_x^\beta u_0(x)| \leq C(1+|x|^2)^{-r}$, with $r > 1$ and $|\beta| \leq s-1$, then the Cauchy problem (1.1) admits a unique global solution $u(x,t)$ satisfying

$$u \in L^\infty([0,\infty); H^s).$$