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THE INTEGRATION OF ALGEBROIDAL FUNCTIONS*

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Abstract In this paper, we introduce the integration of algebroidal functions on Riemann surfaces for the first time. Some properties of integration are obtained. By giving the definition of residues and integral function element, we obtain the condition that the integral is independent of path. At last, we prove that the integral of an irreducible algebroidal function is also an irreducible algebroidal function if all the residues at critical points are zeros.

Key words algebroidal function; integral; integral function element; direct continuation

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1 Introduction

Algebroidal functions are a kind of important multi-value functions. For example, in the field of complex differential equation, algebroidal solutions are more general than meromorphic solutions. But there are few results on algebroidal functions for lack of effective tools. Though integration is a basic definition, we have not seen any research on the integral of algebroidal functions, even in [1, 2]. Note that the integral of algebroid functions we define is completely

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different from [3]. For holomorphic functions, the integrals can be defined as the limit of the same kind as that encountered in a usual integral. But for algebroidal functions, because of its multi-valuedness, we can not define the integral as what we usually do for holomorphic functions. In this paper, we see algebroidal functions as single-valued on the Riemann surfaces. The definite integral on the Riemann surface is defined and some of its properties were obtained in [4, 5]. Then by giving the definition of residues at critical points, we prove that the integral is independent of path if all the residues at critical points are zeros. Under the same assumption, we define the integral function element and prove that they are unchangeable for direct continuation. And last, we obtain the integral of an irreducible algebroidal function is also an irreducible algebroidal function if all the residues at critical points are zeros.

First, we provide the definition of algebroidal functions. Let $A_1(z)$, $A_2(z), \cdots, A_k(z)$ be a group of meromorphic functions in the complex plane $\mathbf{C}$, then the equation in two variables

$$\Psi(W,z)=W^k+A_1(z)W^{k-1}+\cdots+A_k(z)=0 \tag{1.1}$$

defines a k-valued algebroidal function $W(z)$ in $\mathbf{C}$.

Equation (1.1) is irreducible if it can not be expressed as the product of two non-meormorphic functions. In this case, we say the algebroidal function is irreducible. In this paper, we confine our consideration on irreducible algebroidal functions. We use the standard definitions and notations of algebroidal functions, e.g., see [1, 6–12].

The resultant of $\Psi(W,z)$ and its partial derivative $\Psi_W(W,z)$, which is said to be the discrimination of $W(z)$, is denoted by $R(\Psi,\Psi_W)(z)$. For an irreducible algebroidal function, we have $R(\Psi,\Psi_W)(z)\not\equiv 0$. Hence, points in the complex plane can be divided into two kinds, say critical points and regular points. By critical points, we mean points in the set $S_W =: \{z; R(\Psi,\Psi_W)(z)\neq 0\}\cup\{z; z \text{ is the pole of some } A_j(z),\ j=0,1,\cdots,k\}$. And points in the set $T_W =: \mathbf{C}-S_W$ are regular points. It can be deduced that critical points are isolated.

Definition 1.1 By a function element $(p(z),D_a)$, or $(p(z),a)$, we mean a simply-connected domain D_a including $a\in\mathbf{C}$ and a holomorphic function $p(z)$ in D_a. Two function elements $(p(z),a)$ and $(q(z),b)$ are equal, if $a=b$ and there is a neighborhood U of a such that $p(z)=q(z)$ in U. If for all $z\in D_a$, we have $\Psi(z,p(z))=0$, then $(p(z),D_a)$ is said to be a function element of the algebroidal function $W(z)$.

Suppose $W(z)$ is an irreducible algebroidal function defined by (1.1). If there is an ordered pair (w_0,z_0) satisfying

(i) $\Psi(w_0,z_0)=0$;

(ii) $\Psi_W(w_0,z_0)\neq 0$,

by the implicit function theorem, then there uniquely exists a function element $(w(z),D_{z_0})$ such that $w(z_0)=w_0$ and $\Psi(w(z),z)\equiv 0$ for all $z\in D_{z_0}$ (see [5]). And, the ordered pair $(w(z),D_{z_0})$ is a function element of the algebroidal function $W(z)$. We denote the set of all the function elements of the algebroidal function $W(z)$ by $\widetilde{T}_W$.

Definition 1.2 A function element $(q(z),b)$ is said to be the direct continuation of $(p(z),a)=(p(z),D_a)$, if two conditions hold:

(i) $b\in D_a$;

(ii) there is a neighborhood U of b, such that $U\in D_a$ and $p(z)=q(z)$ for all $z\in U$.

Hence, it can be denoted by $(p(z),b)=(q(z),b)$.