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AVERAGE REGULARITY OF THE SOLUTION TO AN EQUATION WITH THE RELATIVISTIC-FREE TRANSPORT OPERATOR*

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Abstract Let $u = u(t, \mathbf{x}, \mathbf{p})$ satisfy the transport equation $\frac{\partial u}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial u}{\partial \mathbf{x}} = f$, where $f = f(t, \mathbf{x}, \mathbf{p})$ belongs to $L^p((0,T) \times \mathbf{R}^3 \times \mathbf{R}^3)$ for $1 < p < \infty$ and $\frac{\partial}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial}{\partial \mathbf{x}}$ is the relativistic-free transport operator from the relativistic Boltzmann equation. We show the regularity of $\int_{\mathbf{R}^3} u(t, \mathbf{x}, \mathbf{p})\mathrm{d}\mathbf{p}$ using the same method as given by Golse, Lions, Perthame and Sentis. This average regularity is considered in terms of fractional Sobolev spaces and it is very useful for the study of the existence of the solution to the Cauchy problem on the relativistic Boltzmann equation.

Key words regularity; transport operator; relativistic Boltzmann equation

2010 MR Subject Classification 76P05; 35Q75

1 Introduction

We are concerned with the average regularity of the solution to an equation with the relativistic-free transport operator from the relativistic Boltzmann equation. Let us begin with the relativistic Boltzmann equation in the following form

$$\frac{\partial u}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial u}{\partial \mathbf{x}} = Q(u, u), \tag{1.1}$$

where $u = u(t, \mathbf{x}, \mathbf{p})$ is a distribution function of a one-particle relativistic gas with the time $t \in (0, \infty)$, the position $\mathbf{x} \in \mathbf{R}^3$, and the momentum $\mathbf{p} \in \mathbf{R}^3$; $p_0 = (1 + |\mathbf{p}|^2)^{1/2}$ denotes the energy of a dimensionless relativistic gas particle with the momentum $\mathbf{p}$; $\frac{\partial}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial}{\partial \mathbf{x}}$ is called the relativistic-free transport operator; $Q(u, u)$ is the relativistic Boltzmann collision operator which can be written as the difference between the gain and loss terms respectively given by Dudyński and Ekiel-Jeżewska [14] in the following forms

$$Q^+(u, u) = \int_{\mathbf{R}^3 \times S^2} \frac{gs^{1/2}}{p_0 p_{*0}} \sigma(g, \theta) u(t, \mathbf{x}, \mathbf{p}') u(t, \mathbf{x}, \mathbf{p}'_*) \mathrm{d}\omega \mathrm{d}\mathbf{p}_*, \tag{1.2}$$

$$Q^-(u, u) = \int_{\mathbf{R}^3 \times S^2} \frac{gs^{1/2}}{p_0 p_{*0}} \sigma(g, \theta) u(t, \mathbf{x}, \mathbf{p}) u(t, \mathbf{x}, \mathbf{p}_*) \mathrm{d}\omega \mathrm{d}\mathbf{p}_*. \tag{1.3}$$

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It is worth mentioning that the gain and loss terms of the relativistic Boltzmann equation can be expressed in other various forms (see [8]). The other different parts in eqs. (1.2) and (1.3) are explained as follows.

$\mathbf{p}$ and $\mathbf{p}_*$ are dimensionless momenta of two relativistic particles immediately before collision while $\mathbf{p}'$ and $\mathbf{p}'_*$ are dimensionless momenta after collision; $p_{*0} = (1+|\mathbf{p}_*|^2)^{1/2}$ denotes the dimensionless energy of the colliding relativistic gas particle with the momentum $\mathbf{p}_*$ before collision, and as used below in the same way, $p'_0 = (1+|\mathbf{p}'|^2)^{1/2}$ and $p'_{*0} = (1+|\mathbf{p}'_*|^2)^{1/2}$ represent the dimensionless energy of the two relativistic particles after collision. $s = |p_{*0} + p_0|^2 - |\mathbf{p}_* + \mathbf{p}|^2$ and $s^{1/2}$ is the total energy in the center-of-mass frame [14]; $g = \sqrt{|\mathbf{p}_* - \mathbf{p}|^2 - |p_{*0} - p_0|^2}/2$ and $2g$ is in fact the value of the relative momentum in the center-of-mass frame [14]; it can be seen that $s = 4 + 4g^2$; $\sigma(g,\theta)$ is the differential scattering cross section of the variable g and the scattering angle θ; $\mathbf{R}^3$ is a three-dimensional Euclidean space and S^2 a unit sphere surface with an infinitesimal element $\mathrm{d}\omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ for the scattering angle $\theta \in [0,\pi]$ and the other solid angle $\varphi \in [0, 2\pi]$ in the center-of-momentum system, and the scattering angle θ is defined by $\cos\theta = 1 - 2[(p_0 - p_{*0})(p_0 - p'_0) - (\mathbf{p} - \mathbf{p}_*)(\mathbf{p} - \mathbf{p}')]/(4 - s)$.

There was a long history of the study of the relativistic Boltzmann equation as one of the most importance in the relativistic kinetic theory. The study of the relativistic kinetic theory began in 1911 when Jüttner [31] derived an equilibrium distribution function of relativistic gases. Lichnerowicz and Marrot [24] were the first to derive the full relativistic Boltzmann equation including the collision operator in 1940. The research of this equation can be roughly classified into four aspects: 1) the derivation of this equation; 2) its relativistic hydrodynamic limit; 3) its Chapman-Enskog approximation and hydrodynamic modes; 4) the existence and uniqueness of the solution to the Cauchy problem on it. For both of 1) and 2), we can see the recent references from Dolan & Zenios [13], Debbasch & Leeuwen [9, 10], Tsumura & Kunihiro [38] and Denicol et al. [11]. For 3), in the early 60's, many researchers, such as Israel [23], applied the Chapman-Enskog expansion into studying the approximative solution to the relativistic Boltzmann equation. The progress of the last research field are recently great. In 1967, Bichteler [5] first proved that the relativistic Boltzmann equation admits a unique local solution under the assumptions that the differential scattering cross-section is bounded and that the initial distribution function decays exponentially with energy. In 1988, Dudyński and Ekiel-Jeżewska [14] proved that the Cauchy problem on the linearized relativistic Boltzmann equation has a unique solution in L^2 space. Four years later, Dudyński and Ekiel-Jeżewska [15] showed that there exists a DiPerna-Lions renormalized solution [12] to the Cauchy problem on the relativistic Boltzmann equation with large initial data in the case of the relativistic soft interactions. Glassey and Strauss [18] proved in 1993 that a unique global smooth solution to this problem exponentially converges to a relativistic Maxwellian as the time goes to infinity if all initial data are periodic in the space variable and near equilibrium. Then in 1995, Glassey and Strauss extended the above result to the whole space case [19] and found that the solution has the property of polynomial convergence with respect to the time. In 1996, Andréasson [1] showed the regularity of the gain term and the strong L^1 convergence to equilibrium for the relativistic Boltzmann equation. Afterward, Jiang gave the global existence of solution to the relativistic Boltzmann equation with hard interactions in the whole space for initial data with finite mass, energy and inertia [27], or in a periodic box for initial data with finite mass and energy [28, 29].