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EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION TO THE INCOMPRESSIBLE NAVIER-STOKES-LANDAU-LIFSHITZ MODEL IN 2-DIMENSION*

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Abstract In this paper, we prove the existence and uniqueness of the weak solution to the incompressible Navier-Stokes-Landau-Lifshitz equations in two-dimension with finite energy. The main techniques is the Faedo-Galerkin approximation and weak compactness theory.

Key words global finite energy weak solution; incompressible Navier-Stokes-Landau-Lifshitz system; Faedo-Galerkin method

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1 Introduction

In this paper, we investigate the incompressible Navier-Stokes-Landau-Lifshitz system (NSLL) in $(0,T)\times\Omega$ [1]:

$$\partial_t\rho+\mathrm{div}(\rho u)=0, \tag{1.1}$$

$$\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla P=\Delta u-\lambda\nabla\cdot(\nabla d\odot\nabla d), \tag{1.2}$$

$$d_t+u\cdot\nabla d+\alpha_1 d\times(d\times\Delta d)=\alpha_2 d\times\Delta d,\quad |d|=1, \tag{1.3}$$

$$\mathrm{div} u=0, \tag{1.4}$$

here we denote by $\Omega\subset\mathbf{R}^2$ the two-dimensional periodic domain, i.e., $\Omega=\{x=(x_1,x_2)||x_i|<D\ (i=1,2)\}$. ρ and u represent the density and the velocity field of the flow, respectively. P is the pressure function. $d(x,t):\Omega\to S^2$, the unit sphere in $\mathbf{R}^3$, denotes the magnetization field. α_2 is a positive constant, and $\alpha_1\geq 0$ is Gilbert damping coefficient. $\nabla\cdot$ denotes the

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divergence operator, and $\nabla d \odot \nabla d$ denotes the 2×2 matrix whose (i, j)-the entry is given by $\nabla_i d \cdot \nabla_j d$ for $1 \leq i, j \leq 2$. For simplicity, we set $\lambda = \alpha_1 = \alpha_2 = 1$.

The initial condition is

$$(\rho_0, u_0, d)|_{t=0} = (\rho_0, u_0, d_0), \quad |d_0| = 1, \tag{1.5}$$

where ρ_0, u_0 and d_0 are 2D-periodic functions.

The system (1.1)–(1.4) is a coupling between the incompressible Navier-Stokes equations and Landau-Lifshitz equation. This model can be used to describe the dispersive theory of magnetization of ferromagnets.

Remark 1.1 If $u = 0$ and ρ is a constant, the system (1.1)–(1.4) becomes the Landau-Lifshitz equation [2].

In 1981, Zhou and Guo proved the existence of the global weak solutions to the initial value problems and initial boundary value problems for Landau-Lifshitz equations from one dimension to multi-dimensions [3]; Alouges and Soyeur [4] proved similar results by penalty method in 1992. However, the regularity and the uniqueness were unsolved in the 1980s due to the complexity of Landau-Lifshitz equations.

However, in 1991, Zhou, Guo and Tan [5] obtained the existence and uniqueness of global smooth solution to one-dimensional Landau-Lifshitz equations with or without Gilbert damping by using a mobile frame on S^2 and some fine a priori estimates.

In 1993, Guo and Hong began the studies on two-dimensional Landau-Lifshitz equations. They established in [6] the relations between two-dimensional Landau-Lifshitz equations and harmonic maps and applied the approaches studying harmonic maps to get the global existence and uniqueness of partially regular weak solution. Later on, in 1998, Chen, Ding and Guo [7] further proved that all the weak solutions with finite energy must be the Chen-Struwe solutions [8]. The uniqueness was also given. This says that the weak solution with finite energy is globally smooth with exception of finitely many singular points at most.

From the beginning of the new century, more and more mathematicians are interested in the researches of Landau-Lifshitz equations. We refer the readers to the works by Guo, Su, Wang, Ding, Carbou and Harpes, et al. [9–16] on Landau-Lifshitz equations and Landau-Lifshitz-Maxwell equations.

A natural question is the regularity of weak solutions to the higher dimensional Landau-Lifshitz equations. In this aspect, in 2004, Liu [17] proved that the "stationary" weak solutions of higher dimensional Landau-Lifshitz equations are partially regular. The Hausdorf dimensions and the Hausdorf measures of the singular set were estimated. These extend the results on harmonic map heat flow by Feldman [18] to Landau-Lifshitz equations. At the same time, Moser [19] obtained the similar results for lower dimensional Landau-Lifshitz equations by different methods.

We know that the "stationary" conditions are hard to verify. So, in 2005, Melcher [20] proved the partial regularity for the weak solutions to the initial value problems of Landau-Lifshitz equations. However, as stated by Melcher, his method does not fit the other dimensional problems and, the partial regularity of weak solutions to the boundary value problems are still unsolved.

This attracted the attention of Changyou Wang at the University of Kentucky. Wang [21],