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HEAT KERNEL ESTIMATES ON JULIA SETS*

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Abstract We give heat kernel estimates on Julia sets $J(f_c)$ for quadratic polynomials $f_c(z) = z^2 + c$ for c in the main cardioid or the $\pm\frac{1}{k}$-bulbs where $k \geq 2$. First we use external ray parametrization to construct a regular, strongly local and conservative Dirichlet form on Julia set. Then we show that this Dirichlet form is a resistance form and the corresponding resistance metric induces the same topology as Euclidean metric. Finally, we give heat kernel estimates under the resistance metric.

Key words Julia sets; heat kernel; resistance metric

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1 Introduction

Analysis on fractals was widely developed. Most of the examples studied are linear fractals such as Sierpinski gasket and Vicsek set. Julia sets are nonlinear fractals but Kigami's paradigm developed in [6] is still applicable. [11] studied the basilica Julia set; [1], [4] and [12] studied more general Julia sets. We use the same method to construct energies and study heat kernel estimates. The main purpose of this paper includes that

- we use external ray parametrization to study Julia sets where the topological property of this parametrization is widely used instead of the linear property used in linear fractals;
- we construct a regular, strongly local and conservative Dirichlet form $(\mathcal{E}, \mathcal{F})$ on Julia set J which is a resistance form and the corresponding resistance metric R induces the same topology as Euclidean metric $|\cdot|$ by verifying the conditions in [11];
- we verify conditions in [8] to obtain heat kernel estimates under the resistance metric in Theorem 7.3 as follows. There exist $C_1, C_2, C_3, C_4 > 0$, $T \in (0, +\infty)$ such that for all $x, y \in J$, $t \in (0, T)$,

$$p_t(x,y) \leq \frac{C_1}{t^{\frac{k}{k+1}}} \exp\left\{-C_2\left(\frac{R(x,y)^{k+1}}{t}\right)^{1/k}\right\},$$

$$p_t(x,y) \geq \frac{C_3}{t^{\frac{k}{k+1}}} \quad \text{if } R(x,y) \leq C_4 t^{\frac{1}{k+1}},$$

where p_t is the heat kernel and k is the period of the attracting cycle of Julia set.

Notation The sign $\asymp$ means that the ratio of the two sides is bounded from above and below by positive constants. The sign $\lesssim$ ($\gtrsim$) means that the LHS is bounded by positive constant times the RHS from above (below).

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2 Julia Sets

Let f be a complex polynomial of degree $d \geq 2$. Given $z \in \mathbb{C}$, the set of iterates $z_0 = z, \cdots, z_{n+1} = f(z_n), \cdots$ is called the forward orbit of z. If there exists a positive integer p such that $f^{\circ p}(z) = z$, then z is called a period point of f and the least p is called the period of z, the orbit of z is called a cycle.

Let z be a periodic point of f with period p, we define multiplier $\lambda = (f^{\circ p})'(z)$. Then z or the cycle of z is called repelling if $|\lambda| > 1$.

Definition 2.1 The filled-in Julia set $K(f)$ of f is defined as

$$K(f) = \left\{z \in \mathbb{C} : f^{\circ n}(z) \nrightarrow \infty \text{ as } n \to +\infty\right\}.$$

The Julia set of f is defined as $J(f) = \partial K(f)$.

The Julia set $J(f)$ of f is the closure of the set of repelling periodic points of f. Moreover, $J(f)$ is connected if and only if the orbits of the critical points of f are bounded (see [3, Page 215–217]).

We have the following properties of Julia sets.

Theorem 2.2 (see [3, Proposition 14.2, Corollary 14.8, Proposition 14.9, 9]) Let f be a polynomial of degree $d \geq 2$, then $J(f)$ is uncountable and compact. If $z \in J(f)$, then

$$J(f) = \overline{\bigcup_{k=1}^{\infty} (f^{-1})^{\circ k}(z)}.$$

We focus on some quadratic polynomials $\{f_c\}_{c \in \mathbb{C}}$, where $f_c(z) = z^2 + c$. The Mandelbrot set M is defined as the set of parameters c for which the Julia set $J(f_c)$ is connected. The Julia sets we consider are the same as those in [4], that is, c in the main cardioid of the Mandelbrot set ($k = 1$) or in a series of bulbs directly adjoining it, namely the $\pm\frac{1}{k}$-bulbs. In the latter part of [4, Page 3917], it is shown that these c satisfy three restrictions given there.

3 External Ray Parametrization

The construction of external rays for polynomial Julia sets was developed in [2] and [10]. Let c belong to a hyperbolic component of the Mandelbrot set and denote $f = f_c$, $J = J(f_c)$, $K = K(f_c)$ for simplicity. Then the Julia set J is connected and locally connected (see [2]).

Since K is compact, there exists a potential function or Green's function $G = G^K : \mathbb{C} \to [0, +\infty)$ defined by

$$G(z) = G^K(z) = \lim_{n \to +\infty} \frac{1}{2^n} \log |f^{\circ n}(z)|.$$

By the definition of filled-in Julia sets, we have the potential function vanishes exactly on K. It is obvious that $G(f(z)) = 2G(z)$ and asymptotic to $\log |z|$ near infinity. We define the external rays of K as the orthogonal trajectories of the level curves $G =$ constant. Each external ray extends to infinity, can be specified by its angle at infinity $\theta \in [0, 2\pi)$, for simplicity, we use parameter $t = \frac{\theta}{2\pi} \in \mathbb{R}/\mathbb{Z}$ and denote this ray by $\mathcal{R}_t^K$. Hence we have a mapping $\phi : \mathbb{S}^1 \to J$, $t \mapsto$ the landing point of $\mathcal{R}_t^K$ where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle. This mapping ϕ is continuous and onto for all c in a hyperbolic component. We say that $x' \in \mathbb{S}^1$ is identified as $x \in J$ if