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CLASSIFICATION OF POSITIVE SOLUTIONS TO A SYSTEM OF HARDY-SOBOLEV TYPE EQUATIONS*

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Abstract In this paper, we are concerned with the following Hardy-Sobolev type system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u(x)=\dfrac{v^q(x)}{|y|^{t_2}} \\ (-\Delta)^{\frac{\alpha}{2}}v(x)=\dfrac{u^p(x)}{|y|^{t_1}}, \end{cases} \quad x=(y,z)\in(\mathbb{R}^k\backslash\{0\})\times\mathbb{R}^{n-k}, \tag{0.1}$$

where $0<\alpha<n$, $0<t_1,t_2<\min\{\alpha,k\}$, and $1<p\le\tau_1:=\frac{n+\alpha-2t_1}{n-\alpha}$, $1<q\le\tau_2:=\frac{n+\alpha-2t_2}{n-\alpha}$. We first establish the equivalence of classical and weak solutions between PDE system (0.1) and the following integral equations (IE) system

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^n}G_\alpha(x,\xi)\frac{v^q(\xi)}{|\eta|^{t_2}}\mathrm{d}\xi \\ v(x)=\displaystyle\int_{\mathbb{R}^n}G_\alpha(x,\xi)\frac{u^p(\xi)}{|\eta|^{t_1}}\mathrm{d}\xi, \end{cases} \tag{0.2}$$

where $G_\alpha(x,\xi)=\frac{c_{n,\alpha}}{|x-\xi|^{n-\alpha}}$ is the Green's function of $(-\Delta)^{\frac{\alpha}{2}}$ in $\mathbb{R}^n$. Then, by the method of moving planes in the integral forms, in the critical case $p=\tau_1$ and $q=\tau_2$, we prove that each pair of nonnegative solutions (u,v) of (0.1) is radially symmetric and monotone decreasing about the origin in $\mathbb{R}^k$ and some point z_0 in $\mathbb{R}^{n-k}$. In the subcritical case $\frac{n-t_1}{p+1}+\frac{n-t_2}{q+1}>n-\alpha$, $1<p\le\tau_1$ and $1<q\le\tau_2$, we derive the nonexistence of nontrivial nonnegative solutions for (0.1).

Key words Hardy-Sobolev type systems; systems of fractional Laplacian; systems of integral equations; method of moving planes in integral forms; radial symmetry; nonexistence

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1 Introduction

Let $n \geq 2$, $1 \leq k < n$, we consider the following Hardy-Sobolev type system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = \dfrac{v^q(x)}{|y|^{t_2}} \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = \dfrac{u^p(x)}{|y|^{t_1}}, \end{cases} \quad x = (y, z) \in (\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{n-k}, \tag{1.1}$$

where $0 < \alpha < n$, $0 < t_1, t_2 < \min\{\alpha, k\}$, and $1 < p \leq \tau_1 := \frac{n+\alpha-2t_1}{n-\alpha}$, $1 < q \leq \tau_2 := \frac{n+\alpha-2t_2}{n-\alpha}$.

In particular, if we set $u = v$, $p = q$ and $t_1 = t_2$, PDE system (1.1) will degenerate into a single Hardy-Sobolev type equation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{u^p(x)}{|y|^t}, \ \ x = (y, z) \in (\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{n-k}. \tag{1.2}$$

Equations of type (1.2) are closely related to the study of the sharp constants of the Hardy-Sobolev inequality and the Caffarelli-Kohn-Nirenberg inequality (see [2, 4, 9, 13, 24, 26] and the references therein). The quantitative and qualitative properties of solutions for equations (1.2) were extensively studied by many authors (see [3–6, 11, 21, 25–28] and the references therein). When $t = 0$, $p = \frac{n+\alpha}{n-\alpha}$, the positive solutions of semi-linear PDEs (1.2) were classified by Lin in [23] for $\alpha = 4$, by Wei and Xu in [28] for even integer $0 < \alpha < n$ and by Chen, Li and Ou in [11] for general real number $0 < \alpha < n$; in the special case $n \geq 3$ and $\alpha = 2$, equation (1.2) becomes the well-known Yamabe problem (for related results, refer to Gidas, Ni and Nirenberg [17], Caffarelli, Gidas and Spruck [1], Chen and Li [7] and Li [22]). In [15], D'Ambrosio, Mitidieri and Pohozaev proved some necessary and sufficient conditions for weak solutions of a class of second order partial differential equations to satisfy the representation formulae and inequalities. If $t > 0$, in the special case $k = n$, when $p = \tau := \frac{n+\alpha-2t}{n-\alpha}$, Lu and Zhu [26] proved that every positive solutions of (1.2) is radially symmetric and monotone decreasing about origin; when $0 < p \leq \frac{n-t}{n-\alpha} < \tau$, Lei [21] obtained the nonexistence of positive solutions for (1.2). If $n \geq 3$, $t > 0$ and $1 \leq k < n$, when $\alpha = 2$, Cao and Li [3] classified the positive solutions of (1.2) for the critical exponent $p = \frac{n+2-2t}{n-2}$ in the special case $t = 1$, they also proved the nonexistence of positive solutions for the subcritical exponent $1 < p < \frac{n+2-2t}{n-2}$. Recently, Chen and Fang [5] generalized Cao and Li's results from $\alpha = 2$ to any real value $\alpha \in (0, n)$ and any $0 < t < \min\{\alpha, k\}$. Our results in this paper extend the classification results for a single equation in [3, 5, 26] to systems of differential equations.

There are also large amounts of literatures devoted to investigating the PDE systems of type (1.1) and related IE systems (see [6, 8, 10, 12, 16, 19, 20, 29, 30] and the references therein). When $t_1 = t_2 = 0$, Chen, Li and Ou proved in [12] that every nonnegative solution pair (u, v) of (1.1) are radially symmetric and decreasing about some point. When $t_1, t_2 < 0$, in the special case $k = n$, Zhang [30] proved that every nonnegative solution pair (u, v) of (1.1) are radially symmetric and decreasing about some point for critical exponents $p = \tau_1$ and $q = \tau_2$, the nonexistence of positive solutions for (1.1) was also obtained in [30] for some subcritical exponents $\frac{n-t_1}{p+1} + \frac{n-t_2}{q+1} > n - \alpha$, $p \in (\frac{n-t_1}{n-\alpha}, \tau_1]$ and $q \in (\frac{n-t_2}{n-\alpha}, \tau_2]$.

In this paper, we will study the classification of nonnegative solutions of Hardy-Sobolev type PDE system (1.1) in the range $t_1, t_2 > 0$ and $1 \leq k < n$ by using the Gidas, Ni and Nirenberg's method of moving planes [17, 18] in integral forms due to Chen, Li and Ou [11].