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LEFT-RIGHT BROWDER LINEAR RELATIONS AND RIESZ PERTURBATIONS*

Teresa ÁLVAREZ
Department of Mathematics, University of Oviedo, 33007, Oviedo, Asturias, Spain
E-mail: *seco@uniovi.es*

Abstract A closed linear relation T in a Banach space X is called left (resp. right) Fredholm if it is upper (resp. lower) semiFredholm and its range (resp. null space) is topologically complemented in X. We say that T is left (resp. right) Browder if it is left (resp. right) Fredholm and has a finite ascent (resp. descent). In this paper, we analyze the stability of the left (resp. right) Fredholm and the left (resp. right) Browder linear relations under commuting Riesz operator perturbations. Recent results of Zivkovic et al. to the case of bounded operators are covered.

Key words left and right Fredholm linear relations; left and right Browder linear relations; Riesz perturbations

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1 Introduction

We adhered to the notations and terminology of [6] and [14]. Let E, F and G be linear spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation A from E to F is any mapping from a linear subspace $D(A)$ of E called the domain of A, into the collection of nonempty subsets of F such that $A(\alpha x_1 + \beta x_2) = \alpha Ax_1 + \beta Ax_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(A)$. For $x \in E$, $x \notin D(A)$ we define $Ax = \emptyset$. With this convention, we have that $D(A) := \{x \in E : Ax \neq \emptyset\}$. The set of all linear relations from E to F is denoted by $LR(E, F)$. An element $A \in LR(E, F)$ is uniquely determined by its graph, $G(A)$, which is defined by

$$G(A) := \{(x, y) \in E \times F : x \in D(A), y \in Ax\},$$

so that, we can identify A with its graph. Moreover, in the sequel, the term "subspace" always refers to a linear subspace.

Let $A \in LR(E, F)$. The inverse of A is the linear relation $A^{-1} \in LR(F, E)$ given by $G(A^{-1}) := \{(y, x) : (x, y) \in G(A)\}$. The subspaces $N(A) := A^{-1}(0)$, $R(A) := A(D(A)) := AD(A)$ and $A(0)$ are called the null space, the range and the multivalued part of A, respectively. We say that A is injective if $N(A) = \{0\}$ and A is called surjective if $R(A) = F$. Observe that A is the graph of an operator if and only if $A(0) = \{0\}$ and the following equalities hold

$$D(A^{-1}) = R(A), \quad R(A^{-1}) = D(A), \quad \text{and} \quad N(A^{-1}) = A(0).$$

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If M is a subspace of E then $A\mid_M$ is given by $G(A\mid_M) := G(A)\cap(M\times F)$ and if $E=F$ then A_M is defined by $G(A_M) := G(A)\cap(M\times M)$.

For linear relations $A,B\in LR(E,F)$ and $C\in LR(F,G)$ and $\lambda\in\mathbb{K}$ the linear relations $A+B$, $A\oplus B$, λA and CA are defined by

$$G(A+B) := \{(x,y+z) : (x,y)\in A, (x,z)\in B\},$$

$$G(A\widehat{+}B) = \{(x+u,y+v) : (x,u)\in A, (y,v)\in B\},$$

this last sum is direct when $G(A)\cap G(B)=\{(0,0)\}$. In such case we write $A\oplus B$,

$$G(\lambda A) := \{(x,\lambda y) : (x,y)\in A\},$$

$$G(CA) := \{(x,z) : (x,y)\in A, (y,z)\in C \ \text{ for some } y\in F\}.$$

Assume that $E=F$ and let $A\in LR(E,E) := LR(E)$, that is, A is a linear relation in E. Then $A-\lambda := A-\lambda I$ where I is the identity operator in E and the resolvent set of A is the set $\rho(A) := \{\lambda\in\mathbb{K} : A-\lambda \text{ is injective and surjective}\}$.

The product of linear relations is clearly associative. Hence A^n, $n\in\mathbb{Z}$, is defined as usual with $A^o := I$ and $A^1 := A$. The hyper-kernel, the hyper-range and the singular chain manifold of $A\in LR(E)$ are defined by

$$N^\infty(A) := \cup_{n\in\mathbb{N}} N(A^n), \qquad R^\infty(A) := \cap_{n\in\mathbb{N}} R(A^n)$$

and

$$R_c(A) := (\cup_{n\in\mathbb{N}} N(A^n))\cap(\cup_{n\in\mathbb{N}} A^n(0)),$$

respectively.

The ascent and the descent of $A\in LR(E)$ are given by

$$a(A) := \min\{p\in\mathbb{N}\cup\{0\} : N(A^p)=N(A^{p+1})\},$$

$$d(A) := \min\{q\in\mathbb{N}\cup\{0\} : R(A^q)=R(A^{q+1})\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and the descent of A are defined to be ∞. Clearly A is injective if and only if $a(A)=0$ and A is surjective if and only if $d(A)=0$. In [14], the authors introduce and give a systematic treatment of these notions of ascent and descent of a linear relation in a linear space. They show that many of the results of Taylor [17] and Kaashoek [11] for operators remain valid in the context of linear relations only under the additional condition $R_c(A)=\{0\}$.

Let X and Y be normed spaces and let $T\in LR(X,Y)$. Then it is easy to see that Q_TT is an operator where Q_T denotes the quotient map $Q_{\overline{T(0)}}$ from Y onto $Y/\overline{T(0)}$. We say that T is closed if its graph is a closed subspace of $X\times Y$, continuous if $\| T \| := \| Q_TT \| < \infty$ and T is called bounded if it is everywhere defined and continuous. Recall that a closed subspace M of X is topologically complemented in X if there exists a closed subspace M_1 of X such that $X=M\oplus M_1$. If this is the case then M_1 is called a topological complement of M in X. In [7] the authors showed that the notion of topological complementation can be studied in terms of multivalued linear projections.

We also adopt the following notation: if M and N are subspaces of X and X' (the dual space of X), then $M^\perp := \{x'\in X' : x'(M)=0\}$ and ${}_\perp N := \{x\in X : N(x)=0\}$. It is know that $\overline{M} = {}_\perp(M^\perp)$ and $({}_\perp N)^\perp$ is the closure of N in the $\sigma(X',X)$- topology in X'.