Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2017,**37B**(5):1483–1496
Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

THE ENERGY FUNCTION WITH RESPECT TO THE ZEROS OF THE EXCEPTIONAL HERMITE POLYNOMIALS*

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Abstract We examine the energy function with respect to the zeros of exceptional Hermite polynomials. The localization of the eigenvalues of the Hessian is given in the general case. In some special arrangements we have a more precise result on the behavior of the energy function. Finally we investigate the energy function with respect to the regular zeros of the exceptional Hermite polynomials.

Key words exceptional Hermite polynomials; system of minimal energy; energy function; partitioned matrices

2010 MR Subject Classification 33E30; 41A05

1 Introduction

Exceptional orthogonal polynomials were introduced recently by Gómez-Ullate, Kamran and Milson (cf. e.g. [1, 2]). Besides the classical ones, exceptional orthogonal polynomials played a fundamental role in the construction of bound-state solutions to exactly solvable potentials in quantum mechanics. The relationship between exceptional orthogonal polynomials and the Darboux transform was observed by Quesne (cf. e.g. [3]). Higher-codimensional families were introduced by Odake and Sasaki [4]. The last few years saw a great deal of activity in the area of exceptional orthogonal polynomials (cf. e.g. [5–8] and the references therein). Exceptional orthogonal polynomials are complete orthogonal systems with respect to a weight on an interval I, and they are also eigenfunctions of a second order differential operator. In all known cases the above mentioned weight is

$$w(x) = \frac{w_0(x)}{P^2(x)},$$

where the classical counterpart of the exceptional polynomials in question are orthogonal with respect to w_0 on I. $P(x)$ is a polynomial with zeros in the exterior of I. The sequence of the degrees of exceptional orthogonal polynomials are gappy, and the number of the missing degrees depends on the degree of P. The zeros of exceptional orthogonal polynomials are divided into

*Received October 8, 2015; revised October 20, 2016. Supported by Hungarian National Foundation for Scientific Research, Grant No. K-100461.

two groups: regular zeros which lie in the domain of orthogonality, and exceptional zeros which lie in the exterior of I. In connection with the location of zeros of exceptional orthogonal polynomials the following conjecture is drafted in [9].

"The regular zeros of exceptional orthogonal polynomials have the same asymptotic behavior as the zeros of their classical counterpart. The exceptional zeros converge to the zeros of $P(x)$."

The location of zeros of exceptional Laguerre and Jacobi polynomials were described by Gómez-Ullate, Marcellán and Milson [10], of exceptional Hermite polynomials by Kuijlaars and Milson, [9]. Now we can draw up the following conjecture.

The normalized counting measure on the scaled regular zeros of exceptional orthogonal polynomials tend in the weak star topology to the equilibrium measure defined by w_0.

We investigate here the energy function at the zeros of exceptional Hermite polynomials.

In 1839, Gauss introduced the energy integral of a measure and the notion of equilibrium measure in $\mathbb{R}^3$. As we are on the complex plane $\mathbb{C}$, here we take into consideration the notion of logarithmic energy (for more general setup see e.g. [11, 12]). Using the notation of [13], let Σ be a subset of $\mathbb{C}$, $\mathcal{M}(\Sigma)$ the set of probability Borel measures on Σ and w is an "admissible" weight function (cf. [13, I. (1.10)]). The weighted energy integral of $\mu \in \mathcal{M}(\Sigma)$ is

$$I_w(\mu) := \iint \log \frac{1}{|z-t|w(z)w(t)} \mathrm{d}\mu(z)\mathrm{d}\mu(t),$$

μ_w is the equilibrium measure for which the energy integral is minimal

$$I_w(\mu_w) = \inf_{\mu \in \mathcal{M}(\Sigma)} I_w(\mu).$$

Restricted our investigation to normalized discrete (n-point counting) measures supported on $u_1, \cdots, u_n \in \Sigma$ the energy integral is

$$I_{w_n}(\mu(u_1, \cdots, u_n)) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \log \frac{1}{|u_i - u_j| w_n(u_i) w_n(u_j)},$$

where instead of w we introduce $w_n := w^{\frac{1}{2(n-1)}}$. This leads to examine the so-called energy function

$$\sup_{u_1, \cdots, u_n \in \Sigma} T_w(u_1, \cdots, u_n) = \prod_{i=1}^{n} w(u_i) \prod_{1 \le i < j \le n} (u_i - u_j)^2.$$

At the end of the nineteenth century the discrete (unweighted) problem was already investigated by some authors e.g. Stieltjes and Hilbert. Few years later Schur proved that on finite or infinite intervals (under certain conditions) the maximum is attained at the zeros of certain classical orthogonal polynomials (cf. [14] and the references therein). The question has both electrostatic and gravitation interpretation. If the set Σ is compact there are n-point systems for which the infimum of the energy is attained. These systems are the Fekete points (cf. [15]) and the discrete n-point energy is the n^{th} transfinite diameter of the set Σ. It is known even in locally compact spaces that the normalized counting measures supported on Fekete sets tend to the equilibrium measure in weak-star sense when n tends to infinity. These systems of minimal energy are expansively examined and have several applications, for instance these are the optimal systems of nodes of interpolation, see e.g. [16–20]. If w is a classical weight function, then the solutions of the weighted energy problem are the sets of the zeros of classical