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BSDES IN GAMES, COUPLED WITH THE VALUE FUNCTIONS. ASSOCIATED NONLOCAL BELLMAN-ISAACS EQUATIONS*

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Abstract We establish a new type of backward stochastic differential equations (BSDEs) connected with stochastic differential games (SDGs), namely, BSDEs strongly coupled with the lower and the upper value functions of SDGs, where the lower and the upper value functions are defined through this BSDE. The existence and the uniqueness theorem and comparison theorem are proved for such equations with the help of an iteration method. We also show that the lower and the upper value functions satisfy the dynamic programming principle. Moreover, we study the associated Hamilton-Jacobi-Bellman-Isaacs (HJB-Isaacs) equations, which are nonlocal, and strongly coupled with the lower and the upper value functions. Using a new method, we characterize the pair (W, U) consisting of the lower and the upper value functions as the unique viscosity solution of our nonlocal HJB-Isaacs equation. Furthermore, the game has a value under the Isaacs' condition.

Key words McKean-Vlasov SDE; BSDE coupled with the lower and the upper value functions; dynamic programming principle; mean-field BSDE; viscosity solution; coupled nonlocal HJB-Isaacs equation; Isaacs' condition

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1 Introduction

Fleming and Souganidis [13] studied two players zero-sum stochastic differential games (SDGs) in a rigorous way. In [13], the authors proved that the lower (resp., upper) value function satisfies the dynamic programming principle (DPP), and is the unique viscosity solution of the associated lower (resp., upper) Hamilton-Jacobi-Bellman-Isaacs (HJB-Isaacs) equation. Moreover, the two HJB-Isaacs equations coincide under Isaacs' condition. Since then, there were many works on SDGs. For example, Buckdahn, Cardaliaguet and Rainer [6] proved

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the existence of Nash equilibrium points for stochastic nonzero-sum differential games with the method introduced in [13]; Bayraktar and Poor [2] discussed stochastic differential games driven by fractional Brownian motion; Browne [5] studied stochastic dynamic investment games in continuous time, and so on. Buckdahn and Li [9] generalized the work of Fleming and Souganidis [13] with a new approach.

On the other hand, mean-field limits can be met in various fields such as quantum chemistry, quantum mechanics, statistical mechanics and physics. In recent years, many authors (see [3, 4, 11, 18, 24, 25]) studied models of large stochastic particle systems with mean-field interaction. From their works, we know that the solution of a linear McKean-Vlasov partial differential equation (PDE) can be interpreted stochastically. In 2009, Buckdahn, Djehiche, Li and Peng [8] got a new type of backward stochastic differential equations (BSDEs), called mean-field BSDEs. Buckdahn, Li, Peng [10] deepened the investigation of mean-field BSDEs, proved the existence and uniqueness theorem as well as the comparison theorem, and investigated the associated nonlocal PDEs. Concerning stochastic maximum principle (SMP) in mean-field type, the reader is referred to Andersson and Djehiche [1], Buckdahn, Djehiche and Li [7], Li [19] and so on.

Generally speaking, DPP for control systems of decoupled mean-field forward-backward stochastic differential equations (FBSDEs) does not hold. But DPP is an important tool to prove the existence of a viscosity solution. When Hao and Li [14] studied the control problems of decoupled mean-field FBSDEs, they met the same difficulty. In order to overcome this difficulty, the authors considered a new type of controlled BSDEs, namely, the BSDEs coupled with the value function, and proved the existence and uniqueness theorem as well as the comparison theorem. Moreover, the authors investigated optimal control problems in the mean-field framework with a new method.

Our paper studies a 2-person zero-sum stochastic differential game, in which the sequential pay-off functional is modeled by a BSDE coupled with the lower value function and the upper value function. As long as one doesn't suppose Isaacs' condition, the lower and the upper value function of the game don't coincide necessarily, but both are the solution of a Hamilton-Jacobi-Bellman-Isaacs equation, which, in the classical case, are not coupled. In our work we look for a model of a stochastic differential game, which makes that these both PDEs become fully coupled. This was obtained by considering sequential cost functionals described by a BSDE, depending itself on the upper and the lower value function. With such BSDEs one can model, for instance, the fact that a difference between the upper and the lower value functions leads "by anticipation" to a higher cost for the disadvantaged player and to a higher pay-off for the advantaged one.

In this paper we study the BSDEs coupled with the lower and the upper value functions, and related stochastic differential games. More precisely, by $\mathcal{U}_{t,T}$ (resp., $\mathcal{V}_{t,T}$) we denote the set of all admissible controls defined on $[t,T]$ and taking values in a compact metric space U (resp., V). $\mathcal{A}_{t,T}$ (resp., $\mathcal{B}_{t,T}$) denotes the set of all nonanticipative strategies $\alpha: \mathcal{V}_{t,s}\to\mathcal{U}_{t,s}$ (resp., $\beta: \mathcal{U}_{t,s}\to\mathcal{V}_{t,s}$). For given $x_0\in\mathbb{R}^n$, $\bar{u}\in\mathcal{U}_{0,T}$, $\bar{v}\in\mathcal{V}_{0,T}$, we consider the following BSDE coupled with the lower and the upper value functions

$$Y_s^{t,x;u,v}=E[\Phi(X_T^{0,x_0;\bar{u},\bar{v}},x)]\Big|_{x=X_T^{t,x;u,v}}+\int_s^T E[f(r,X_r^{0,x_0;\bar{u},\bar{v}},W(r,X_r^{0,x_0;\bar{u},\bar{v}}),$$