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RENORMALIZED SOLUTIONS OF ELLIPTIC EQUATIONS WITH ROBIN BOUNDARY CONDITIONS*

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Abstract In the present paper, we consider elliptic equations with nonlinear and nonhomogeneous Robin boundary conditions of the type

$$\begin{cases} -\operatorname{div}(B(x,u)\nabla u) = f \quad \text{in} \ \ \Omega, \\ u = 0 \ \ \text{on} \ \ \Gamma_0, \\ B(x,u)\nabla u \cdot \vec{n} + \gamma(x)h(u) = g \ \ \text{on} \ \ \Gamma_1, \end{cases}$$

where f and g are the element of $L^1(\Omega)$ and $L^1(\Gamma_1)$, respectively. We define a notion of renormalized solution and we prove the existence of a solution. Under additional assumptions on the matrix field B we show that the renormalized solution is unique.

Key words elliptic equations; renormalized solution; uniqueness; Robin boundary conditions; L^1 data

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1 Introduction

This paper is devoted to the class of nonlinear equations of the type

$$\begin{cases} -\operatorname{div}(B(x,u)\nabla u) = f \quad \text{in} \ \ \Omega, \\ u = 0 \ \ \text{on} \ \ \Gamma_0, \\ B(x,u)\nabla u \cdot \vec{n} + \gamma(x)h(u) = g \ \ \text{on} \ \ \Gamma_1. \end{cases} \tag{1.1}$$

Here Ω is a perforated domain of $\mathbb{R}^N$ $(N \geq 2)$, the matrix field $B(x,s)$ is coercive but does not verify any growth condition with respect to the second variable and f belongs to $L^1(\Omega)$. We impose Dirichlet boundary conditions on Γ_0 which is the outside boundary of Ω and nonlinear and nonhomogeneous Robin conditions on Γ_1 which is the boundary of the holes. Moreover the

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function g belongs to $L^1(\Gamma_1)$, γ is a nonnegative function lying in $L^\infty(\Gamma_1)$, h is a nondecreasing continuous function such that $h(0) = 0$ and $\vec{n}$ is the outer unit normal to $\partial\Omega$.

The main difficulties in dealing with the existence and the uniqueness of the solution to (1.1) are due to the L^1 data f and g, the fact that $B(x, s)$ and $h(s)$ do not verify any growth condition with respect to s and the Robin boundary condition on Γ_1 that requires a specific notion of trace on Γ_1. Problem (1.1) is motivated by the homogenization in a periodically perforated domain.

When f belongs to $L^2(\Omega)$, g belongs to $L^2(\Gamma_1)$ and under some growth conditions on the function h, the existence and uniqueness of a weak solution to (1.1) were studied in [9] and the homogenization of (1.1) with oscillating coefficients in a periodically perforated domain was performed in [10].

When the data f and g belong to $L^1(\Omega)$ and $L^1(\Gamma_1)$, respectively, it is well known that we cannot use the notion of weak solution. Indeed if $B(x, s)$ and $h(s)$ are bounded functions we can expect to have a solution in the sense of distribution by adapting the method of [6] but we do not have in general the uniqueness of such a solution (see the counterexample in [24]). Without any growth condition on B and h we cannot expect to have a solution in the sense of distribution, the field $B(x, u)$ could not belong to $L^1_{\text{loc}}(\Omega)$ for example. In the present paper we use the framework of renormalized solutions. This notion was introduced by DiPerna and Lions in [16] for the first order equations and has been developed for elliptic problems with Dirichlet boundary conditions (i.e., $\Gamma_1 = \emptyset$) and with $L^1(\Omega)$ data in [20] (see also [19]). In [13] the authors gave a definition of a renormalized solution for elliptic problems with general measure data and proved the existence of such a solution. Observe that for elliptic equations with boundary Dirichlet conditions and L^1 data, this notion is equivalent to the notion of entropy solutions (see [4]) and to the notion of solutions obtained as limit of approximations (see [14]). As far as elliptic equations with L^1 data and Dirichlet boundary conditions are concerned we refer to [4, 6, 13, 14, 19, 20] among a wide literature.

Similar equations to (1.1) with Neumann boundary conditions or Robin boundary conditions were studied in [1, 2, 8, 21, 23] using the framework of entropy solution while in [17] the author used a duality method for linear problems with Neumann and Robin boundary conditions. In particular in [1] (see also [2]) the authors studied the equation

$$\begin{cases} u - \operatorname{div}(a(x, \nabla u)) = f & \text{in } \Omega, \\ -a(x, \nabla u) \cdot \vec{n} \in \beta(u) & \text{on } \partial\Omega, \end{cases}$$

where f belongs to $L^1(\Omega)$ and where β is a maximal monotone graph. Using the notion of entropy solution and by defining an appropriate notion of trace (since we cannot expect to have u in a Sobolev space), they prove the existence and the uniqueness of the entropy solution.

In the present paper we give a definition of renormalized solution for equation (1.1) (see Definition 2.6). Since Robin condition requires the trace of u on the boundary Γ_1 in same spirit than [1], we define a notion of trace for functions whose truncates belong to $H^1_{\Gamma_0}$ and verify an additional property. We prove the existence of a renormalized solution of problem (1.1) in Theorem 3.1 under general assumption on B and h. Another interesting question is the uniqueness of such a renormalized solution. Here the operator $u \mapsto -\operatorname{div}(B(x, u)\nabla u)$ is not monotone and it is worth noting that equation (1.1) does not contain any local term in u