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GEVREY REGULARITY WITH WEIGHT FOR INCOMPRESSIBLE EULER EQUATION IN THE HALF PLANE*

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Abstract In this work we prove the weighted Gevrey regularity of solutions to the incompressible Euler equation with initial data decaying polynomially at infinity. This is motivated by the well-posedness problem of vertical boundary layer equation for fast rotating fluid. The method presented here is based on the basic weighted L^2-estimate, and the main difficulty arises from the estimate on the pressure term due to the appearance of weight function.

Key words Gevrey class regularity; incompressible Euler equation; weighted Sobolev space

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1 Introduction

In this paper we study the Gevrey propagation of solutions to incompressible Euler equation. Gevrey class is a stronger concept than the C^∞-smoothness. In fact it is an intermediate space between analytic space and C^∞ space. There have been extensive mathematical investigations (cf. [2, 7, 8, 14–16, 9–11, 13] for instance and the references therein) on Euler equation in different kinds of frames, such as the Sobolev space, the analytic space and the Gevrey space. In this paper we will consider the problems of Gevrey regularity with weight, which is motivated by the study of the vertical boundary layer problem. From a physical point of view, as well as from a mathematical point of view, when the direction of rotation is perpendicular to the boundaries, the boundary equation is well developed and called Ekman layers. Up to now

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the Ekman layers (horizontal layer) are well understood (cf. [1] for instance and the references therein). When the direction of rotation is parallel to the boundaries, the situation is, however, different from the perpendicular case above: the vertical layers are very different and much more complicated, from a physical, analytical and mathematical point of view, and many open questions in all these directions remain open. We refer to [1] for detailed discussion on the vertical layers. Recently we try to investigate the well-posedness problem for vertical layer, and the related and preliminary step is to establish the Gevrey regularty with weight for the outer flow which is described by Euler equation. This is the main result of the present paper.

Without loss of generality, we consider the incompressible Euler equation in half plane $\mathbb{R}^2_+$, where $\mathbb{R}^2_+ = \{(x,y); x\in\mathbb{R}, y\in\mathbb{R}^+\}$, and our results can be generalized to 3-D Euler equation. The velocity $(u(t,x,y), v(t,x,y))$ and the pressure $p(t,x,y)$ satisfy the following equation:

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2_+, \tag{1.1}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2_+, \tag{1.2}$$

$$\partial_x u + \partial_y v = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2_+. \tag{1.3}$$

The boundary condition

$$v\big|_{y=0} = 0 \quad \text{in } (0,\infty)\times\mathbb{R}; \quad u, v\to 0, \text{ as } \sqrt{x^2+y^2}\to\infty. \tag{1.4}$$

With initial data

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad \text{in } \mathbb{R}^2_+. \tag{1.5}$$

Here the initial data (u_0, v_0) satisfy the compatibility condition:

$$\partial_x u_0 + \partial_y v_0 = 0, \quad v_0|_{y=0} = 0, \quad u_0, v_0 \to 0, \quad \text{as} \quad \sqrt{x^2+y^2}\to\infty.$$

Before stating our main result we first introduce the (global) weighted Gevrey space.

Definition 1.1 Let $\ell_x, \ell_y \ge 0$ be real constants that independent of x, y, we say that $f \in G^s_{\tau,\ell_x,\ell_y}(\mathbb{R}^2_+)$ if

$$\sup_{|\alpha|\ge 0} \frac{\tau^\alpha}{|\alpha|!^s} \left\| \langle x\rangle^{\ell_x} \langle y\rangle^{\ell_y} \partial^\alpha f \right\|_{L^2(\mathbb{R}^2_+)} < \infty,$$

where and throughout the paper we use the notation $\langle\cdot\rangle = (1+|\cdot|^2)^{\frac12}$.

In this paper we present the persistence of weighted Gevrey class regularity of the solution, i.e., we prove that if the initial datum (u_0, v_0) is in some weighted Gevrey space and satisfy the compatible condition, then the global solution belongs to the same space. With only minor changes, these results can also be extended to 3-D Euler equation, and the global solution here will be replaced by a local solution, precisely.

Theorem 1.2 Suppose the initial data $u_0 \in G^s_{\tau_0,0,\ell_y}, v_0 \in G^s_{\tau_0,\ell_x,0}$ for some $s\ge 1, \tau_0 > 0$ and $0 \le \ell_x, \ell_y \le 1$. Then the Euler equation (1.1)–(1.5) admits a unique solution (u, v, p) such that

$$u(t,\cdot) \in L^\infty\left([0,+\infty);\ G^s_{\tau(t),0,\ell_y}\right), \quad v(t,\cdot)\in L^\infty\left([0,+\infty);\ G^s_{\tau(t),\ell_x,0}\right),$$

and p satisfies

$$\partial_x p(t,\cdot) \in L^\infty\left([0,+\infty);\ G^s_{\tau(t),0,\ell_y}\right), \quad \partial_y p(t,\cdot)\in L^\infty\left([0,+\infty);\ G^s_{\tau(t),\ell_x,0}\right),$$

where $\tau(t)$ is a decreasing function of t with initial value τ_0.