Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2017,**37B**(3):573–583
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

# GLOBAL WEAK SOLUTIONS TO ONE-DIMENSIONAL COMPRESSIBLE VISCOUS HYDRODYNAMIC EQUATIONS*

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**Abstract** In this article, we are concerned with the global weak solutions to the 1D compressible viscous hydrodynamic equations with dispersion correction $\delta^2\rho((\varphi(\rho))_{xx}\varphi'(\rho))_x$ with $\varphi(\rho)=\rho^\alpha$. The model consists of viscous stabilizations because of quantum Fokker-Planck operator in the Wigner equation and is supplemented with periodic boundary and initial conditions. The diffusion term $\varepsilon u_{xx}$ in the momentum equation may be interpreted as a classical conservative friction term because of particle interactions. We extend the existence result in [1] ($\alpha=\frac{1}{2}$) to $0<\alpha\leqslant 1$. In addition, we perform the limit $\varepsilon\to 0$ with respect to $0<\alpha\leqslant\frac{1}{2}$.

**Key words** Viscous hydrodynamic equations; global weak solution; dispersion correction; periodic boundary and initial conditions

**2010 MR Subject Classification** 35B40; 35G25; 35Q72; 85A30

## 1 Introduction

Diffusive corrections are of great interest in quantum models, which are applied in semiconductor structures. Equations for a dissipative quantum system related to quantum Brownian motion were derived in [2, 3]. This approach leads to a Wigner equation with Fokker-Planck operator in [4]. When particle interactions are taken into account, a quantum Fokker-Planck interaction operator together with the Wigner equation leads to second-order derivative terms (which are viscous terms). In [5–7], because of quantum interactions, a quantum viscous perturbation was derived, and one can refer to [8, 9] for the stationary quantum-regularized model with a classical mass-conservative viscous effect. This yields various viscous quantum hydrodynamic equations.

The aim of this article is to study the following viscous hydrodynamic system:

$$\rho_t+(\rho u)_x=\nu\rho_{xx}, \tag{1.1}$$

*Received March 31, 2016. This work was supported by NSF (11271052).

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \delta^2 \rho((\varphi(\rho))_{xx}\varphi'(\rho))_x = \nu(\rho u)_{xx} + \varepsilon u_{xx} - \frac{\rho u}{\tau}. \tag{1.2}$$

where $\rho$ is density, $u$ is velocity, $\nu$ is the viscosity depending also on the parameter $\delta$, $\varepsilon$ accounts for classical mass-conservative viscous effects, $\tau$ is the relaxation time, $\delta$ is a positive parameter, $p(\rho)$ denotes the pressure, and the dispersion term $\varphi(\rho) = \rho^\alpha$, $0 < \alpha \leqslant 1$. We consider the initial value problem of the system (1.1)–(1.2) in one-dimensional torus $\mathbb{T}$ with initial data:

$$\rho(x,0) = \rho_0(x),\ \ \rho u(x,0) = \rho_0 u_0. \tag{1.3}$$

In the absence of viscous and dispersion effects, that is, $\nu = \varepsilon = \delta = 0$, the above equations represent the hydrodynamic semiconductor equations [10]. When viscous effects are absent, $\nu = \varepsilon = 0$, we obtain different hydrodynamic equations depending on the choices of dispersion term $\varphi(\rho) = \rho^\alpha$ and the positive parameter $\delta$. Letting $\alpha = \frac{1}{2}$ and $\delta$ be the scaled Planck constant, we deduce the quantum hydrodynamic equations; for physical background, refer to [11–14]. When $\alpha = 1$, $\nu = 0$, and $\delta^2 = \tilde{\delta}$ denotes capillary coefficient, Korteweg model [15] was derived.

A moment method applied to Wigner-Fokker-Planck equations leads to the viscous quantum hydrodynamic models for $\varepsilon = 0$ and $\alpha = \frac{1}{2}$. The viscous terms $\nu\rho_{xx}$ and $\nu(\rho u)_{xx}$ in the compressible fluid system (1.1)-(1.2) arise from the Fokker-Planck interaction operator $Q_1(w) = \nu w_{xx}$ in the Wigner equation. It is possible to obtain the diffusive velocity term $\varepsilon u_{xx}$ in the momentum equation (1.2) from the Wigner equation by introducing the heuristic interaction operator $Q_2(w) = \varepsilon \partial_x^2\Big(\frac{w}{\int_{\mathbb{R}} w dk}\Big)$. For more details, see [1].

For $\varepsilon = 0$, $\alpha = \frac{1}{2}$, and $p(\rho) = \rho^\gamma$, Ansgar Jüngel [1] proved the global existence of weak solutions to the corresponding viscous quantum Euler system, which can be reformulated as the quantum Navier-Stokes equations by means of so-called effective velocity. In [16], the global existence of weak solutions and asymptotic limit for the compressible Euler equations with degenerate viscosities and capillarity were investigated, where they used the approximate system with the dispersion term $\delta^2 \rho \nabla(\triangle\varphi(\rho)\varphi'(\rho))$, and this dispersion term was also involved in the diffusive capillary models in [17]. The system (1.1)–(1.2) ($\varepsilon = 0$ and $\alpha = \frac{1}{2}$) without the relaxation-time term and the dispersion term ($\delta = 0$) is sometimes employed as a viscous approximation of the (one-dimensional) Euler equations in the vanishing viscosity method [18].

There are only a few mathematical results for the quantum hydrodynamic model ($\alpha = \frac{1}{2}$) because of the third-order derivatives in the dispersion term; refer to [11, 19–22].

By the energy defined as follows, we obtain a priori bounds. Let $h$ be a function defined by $h'(y) = p'(y)/y$ for $y > 0$ and $h(1) = 0$. $H$ is a primitive of $h$. The energy is given by

$$E(\rho, u) = \int_{\mathbb{T}} \left( H(\rho) + \frac{1}{2}\rho u^2 + \frac{\delta^2}{2}(\varphi(\rho))_x^2 \right) \mathrm{d}x. \tag{1.4}$$

Then, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \nu \int_{\mathbb{T}} (\rho u_x^2 + \delta^2 (\varphi(\rho))_{xx}^2)\mathrm{d}x + \varepsilon \int_{\mathbb{T}} u_x^2 \mathrm{d}x \leqslant K, \tag{1.5}$$

where the constant $K > 0$ depends only on $\nu$. Therefore, we obtain $H^2$ estimates for $\sqrt{\rho}$, and $L^2$ estimates for $u_x$, for fixed $\varepsilon > 0$, which is needed to obtain the lower bound for $\rho$.

In this article, we can extend the existence result in [1] ($\alpha = \frac{1}{2}$) to $0 < \alpha \leqslant 1$. Moreover, we perform the limit $\varepsilon \to 0$ with respect to $0 < \alpha \leqslant \frac{1}{2}$. Here, we only consider the one-dimensional equations because of the key point $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, which is valid only in one dimension.