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SIGN-CHANGING SOLUTIONS FOR p-BIHARMONIC EQUATIONS WITH HARDY POTENTIAL IN $\mathbb{R}^N$*

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Abstract In this article, by using the method of invariant sets of descending flow, we obtain the existence of sign-changing solutions of p-biharmonic equations with Hardy potential in $\mathbb{R}^N$.

Key words Sign-changing solutions; p-biharmonic equations; Hardy potential

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1 Introduction

In this article, we look for $u \in D$ satisfying

$$\int_{\mathbb{R}^N} |\Delta u|^{p-2}\Delta u \Delta\varphi \mathrm{d}x - \int_{\mathbb{R}^N} \frac{\mu}{|x|^{2p}}|u|^{p-2}u\varphi \mathrm{d}x = \int_{\mathbb{R}^N} a(x)|u|^{r-2}u\varphi \mathrm{d}x,\ \forall \varphi \in D, \tag{1.1}$$

where D is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x\Big)^{\frac{1}{p}};$$

$1 < p < \frac{N}{2}$, $p < r < p^* = \frac{Np}{N-2p}$, and $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an operator of fourth order, the so-called p-biharmonic operator. The function a satisfies the condition (A):

(A) $a(x) > 0$ and $a(x) \in L^q(\mathbb{R}^N)$ satisfying $\frac{1}{q} + \frac{r}{p^*} = 1$.

The following Rellich inequality (cf. [1]) will be used in this article:

$$\int_{\mathbb{R}^N} |\Delta u|^p \mathrm{d}x \geq \mu_{N,P} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \mathrm{d}x,$$

where the best constant

$$\mu_{N,p} := \Big(\frac{(p-1)N(N-2p)}{p^2}\Big)^p.$$

Here is the main result of this article.

Theorem 1.1 For each $0 < \mu < \mu_{N,p}$, the problem (1.1) has a positive solution, a negative solution, and a sequence of sign-changing solutions.

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Recently, for more results about the existence of nontrivial solutions and sign-changing solutions of biharmonic equations, see, for example, [2–6], where [6] seems to be one of the first articles on sign-changing solutions for biharmonic boundary value problems. There are many results about biharmonic equations with Hardy potential (for example, [7, 8]). In [8], the existence of one sign-changing solution for unbounded domain was obtained. For nonlinear differential equations involving a general p-biharmonic operator (in particular, p-biharmonic operator) under Dirichlet boundary conditions or Navier boundary conditions, the existence and multiplicity of weak solutions was proved in [9]. In [10] the existence of sign-changing solutions about p-biharmonic equations with Hardy potential in the bounded domain was obtained. Meanwhile in [11], a Navier boundary value p-biharmonic problem on the bounded domain was treated, where the equation does not involve the Hardy potential. In addition, the authors of [12, 13] dealt with a class of inhomogeneous elliptic problems involving the biharmonic operator, where the nonlinear terms are singular continuous functions, and they proved compact embedding results and some existence results. We also point out that the extremal solution for some exponential and power nonlinearities was obtained using an improved weighted Hardy inequality (see [14]), and nonlinear potential theory in variable exponent Sobolev spaces was studied in [15].

In this article, we will consider the p-biharmonic equation with Hardy potential in the whole space $\mathbb{R}^N$. Our results improve those of such as [10] greatly. We will prove Theorem 1.1 using the method of invariant sets of gradient flows (see [16]). In the case that the domain is unbounded, we encounter difficulties caused by the lack of compactness. Another of the key points is to construct a vector field A, which keeps the positive and negative cones invariant. Compared to the elliptic equation of second order, the biharmonic equations have its own difficulties. For example, given $u \in D$, in general, u^+ and u^- do not belong to D. Consequently, constructing the operator A (see Section 3 for details) and verifying its properties need special handling.

This article is organized as follows. In Section 2, we verify the (PS) condition. In Section 3, we construct the operator A and discuss its properties. In Section 4, we prove that problem (1.1) has a positive solution, a negative solution, and a sequence of sign-changing solutions.

2 Verification of the (PS) Condition

We assume that $0 < \mu < \mu_{N,p}$. Now, we define a functional on D by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N}\Big(|\Delta u|^p - \frac{\mu}{|x|^{2p}}|u|^p\Big)\mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u|^r\mathrm{d}x.$$

Clearly, I is of class C^1 and

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^N}\Big(|\Delta u|^{p-2}\Delta u\Delta\varphi - \frac{\mu}{|x|^{2p}}|u|^{p-2}u\varphi\Big)\mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u|^{r-2}u\varphi\mathrm{d}x,\ \forall\varphi\in D.$$

Lemma 2.1 If $u_n \rightharpoonup u$ in D, then, $\int_{\mathbb{R}^N} a(x)|u_n - u|^r\mathrm{d}x \to 0$ as $n\to\infty$.

Proof Denote

$$A = \Big\{x\in\mathbb{R}^N \Big|\, |x| \le R,\, a(x)\le M\Big\}.$$

Then,

$$\int_{\mathbb{R}^N} a(x)|u_n - u|^r\mathrm{d}x = \int_{\mathbb{R}^N\setminus A} a(x)|u_n - u|^r\mathrm{d}x + \int_A a(x)|u_n - u|^r\mathrm{d}x$$