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SELF-SIMILAR SOLUTIONS TO THE HYPERBOLIC MEAN CURVATURE FLOW*

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Abstract This article concerns the self-similar solutions to the hyperbolic mean curvature flow (HMCF) for plane curves, which is proposed by Kong, Liu, and Wang and relates to an earlier proposal for general flows by LeFloch and Smoczyk. We prove that all curves immersed in the plane which move in a self-similar manner under the HMCF are straight lines and circles. Moreover, it is found that a circle can either expand to a larger one and then converge to a point, or shrink directly and converge to a point, where the curvature approaches to infinity.

Key words Hyperbolic mean curvature flow; self-similar solutions; curvature

2010 MR Subject Classification 35L10; 35C06

1 Introduction

In 2009, Kong, Liu, and Wang [6, 7] proposed the hyperbolic mean curvature flow (HMCF) for plane curves. More precisely, they considered the following initial value problem:

$$\begin{cases} \dfrac{\partial^2 F}{\partial t^2}(z,t) = k(z,t)N(z,t) + \rho(z,t)T(z,t), \ \forall\,(z,t) \in \mathbb{R}\times[0,T), \\ F(z,0) = F_0(z), \quad \dfrac{\partial F}{\partial t}(z,0) = h(z)N_0(z), \end{cases} \tag{1.1}$$

where $F(z,t)$ denotes the unknown vector-valued function standing for the curve at time t, $k(z,t)$ the mean curvature of $F(z,t)$, $N(z,t)$ the unit normal vector of $F(z,t)$, $T(z,t)$ the unit tangent vector of $F(z,t)$, and $F_0(z)$ the initial curve; while $h(z)$ and $N_0(z)$ are the magnitude of initial velocity and unit normal vector of initial curve $F_0(z)$, respectively; the function $\rho(z,t)$ is defined by

$$\rho(z,t) = -\left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle, \tag{1.2}$$

in which s is the arclength parameter. The flow described by (1.1) is always normal, that is, the velocity filed is perpendicular to the curve during the evolution. Kong, Liu, and Wang [6] investigated closed plane curves under the HMCF. They found that system (1.1) can be reduced into a hyperbolic Monge-Ampère equation for the support function and showed that

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either the curve converges to a point, or shocks and other singularities are generated in finite time. Moreover, Kong and Wang [7] considered the motion of periodic plane curves under the HMCF. It was proved that if the total variation of initial curve is small enough in one period and some certain condition is satisfied, then singularities must develop in finite time and the lifespan was also given. Recently, Wang [9] extended the above discussions on HMCF to the case in Minkowski space, wherein the motion of immersed spacelike closed curves and the corresponding singularities were investigated.

In this article, we study the self-similar solutions to the HMCF for plane curves. More precisely, we consider the following immersed plane curve:

$$F(z,t) = g(t)e^{\mathrm{i}f(t)}F_0(z) + H(t) \quad \forall\,(z,t) \in \mathbb{R} \times I, \tag{1.3}$$

which is a complex valued function. Here, $F_0(z) : \mathbb{R} \to \mathbb{C}$ is the immersed initial curve and is assumed to be nonzero; $f, g : I \to \mathbb{R}$ and $H : I \to \mathbb{C}$ are differentiable with $f(0) = 0, g(0) = 1$, and $H(0) = 0$. Obviously, the function f determines the rotation, g relates to the scaling, and H is the translation term. We shall show that all self-similar solutions evolving under the HMCF are either straight lines or circles. Moreover, one immersed circle always shrinks into a point, whenever the initial velocity vanishes or not. In addition, the time when a circle converges to a point is derived explicitly. These results presumably are reasonable because the HMCF is a normal one, however, the proof seems nontrivial.

The geometric flow has been investigated for many years and can be dated back at least to Gage and Hamilton [2], wherein a curve shortening flow was studied. They showed that under the curve shortening flow, a strictly convex plane curve must shrink to a unit circle in a certain sense. Recently, Halldorsson [3] concerned all self-similar solutions to the curve shortening flow and gave a complete description and classification. It was shown that apart from straight lines and circles, there are many other immersed curves in the plane under this flow. Huisken and Ilmanen [5] developed a theory of the inverse mean curvature flow and succeeded in proving the Riemann Penrose inequality in general relativity.

For hyperbolic version of geometric flows, He, Kong, and Liu [4] introduced the following hyperbolic mean curvature flow

$$\frac{\partial^2 X}{\partial t^2}(u,t) = H(u,t)N(u,t), \quad \forall\, u \in \mathcal{M}, \ \ \forall\, t > 0,$$

where $\mathcal{M}$ is a Riemannian manifold, $X(\cdot,t) : \ \mathcal{M} \to \mathbb{R}^{n+1}$, and H is the mean curvature and N is the unit inner normal vector. He, Kong, and Liu [4] derived a system of hyperbolic equations and obtained a uniquely short-time smooth solution. Some nonlinear wave equations for curvatures are also provided.

Recently, Lefloch and Smoczyk [8] proposed an interesting hyperbolic geometric evolution equations, which describe the motion of a hypersurface along the direction of its mean curvature vector. The governing equations read

$$\frac{\partial^2}{\partial t^2}X = eH(u,t)\vec{N} - \nabla e, \quad \left(\frac{\partial X}{\partial t}\right)^{\top}_{|t=0} = 0,$$

where the scalar H is the mean curvature of the hypersurface and the vector $\vec{N}$ denotes its unit normal, $e = \frac{1}{2}(|\frac{\mathrm{d}}{\mathrm{d}t}X|^2 + n)$ is the local energy density. The second equation in the above equations means that the tangential part of initial velocity vanishes. They showed that this