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CHARACTERIZATION OF DERIVATIONS ON $\mathcal{B}(X)$ BY LOCAL ACTIONS*

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Abstract Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. A linear map $\delta:\mathcal{A}\to\mathcal{M}$ is said to be Jordan derivable at a nontrivial idempotent $P\in\mathcal{A}$ if $\delta(A)\circ B+A\circ\delta(B)=\delta(A\circ B)$ for any $A,B\in\mathcal{A}$ with $A\circ B=P$, here $A\circ B=AB+BA$ is the usual Jordan product. In this article, we show that if $\mathcal{A}=Alg\mathcal{N}$ is a Hilbert space nest algebra and $\mathcal{M}=\mathcal{B}(H)$, or $\mathcal{A}=\mathcal{M}=\mathcal{B}(X)$, then, a linear map $\delta:\mathcal{A}\to\mathcal{M}$ is Jordan derivable at a nontrivial projection $P\in\mathcal{N}$ or an arbitrary but fixed nontrivial idempotent $P\in\mathcal{B}(X)$ if and only if it is a derivation. New equivalent characterization of derivations on these operator algebras was obtained.

Key words Derivations; triangular algebras; subspace lattice algebras; Jordan derivable maps

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1 Introduction

Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Recall that a linear map $\delta:\mathcal{A}\to\mathcal{M}$ is called a derivation if $\delta(AB)=\delta(A)B+A\delta(B)$ for all $A,B\in\mathcal{A}$; a Jordan derivation if $\delta(A\circ B)=\delta(A)\circ B+A\circ\delta(B)$ for all $A,B\in\mathcal{A}$, here $A\circ B=AB+BA$ is the usual Jordan product. Clearly, each derivation is a Jordan derivation, but the reverse is not true in general. The question of revealing the relations between derivations and Jordan derivations has received considerable attention from many mathematicians. Starting with the results by Herstein [1], Jacobson and Rickart [2], this problem has been an active area of research for more than sixty years.

Recently many mathematicians study derivations and Jordan derivations by their local actions. One popular direction is local mappings problems such as local derivations and 2-local derivations. A linear map $\delta:\mathcal{A}\to\mathcal{M}$ is called a local derivation (2-local derivation), if for each $A\in\mathcal{A}$ ($A,B\in\mathcal{A}$), there exists a derivation δ_A ($\delta_{A,B}$) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A)=\delta_A(A)$ ($\delta(A)=\delta_{A,B}(A)$, $\delta(B)=\delta_{A,B}(B)$). Ayupov and Kudaybergenov studied local derivations and 2-local derivations on finite dimensional Lie algebras (see [3–7] and references therein). The other direction is to study linear maps that are derivable or Jordan derivable at some fixed point.

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Assume that $\delta : \mathcal{A} \to \mathcal{M}$ is a linear map, following [8–13], δ is said to be derivable (Jordan derivable) at a given point $Z \in \mathcal{A}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ ($\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$) for any $A, B \in \mathcal{A}$ with $AB = Z$. It is obvious that a linear map is a derivation (Jordan derivation) if and only if it is derivable (Jordan derivable) at every point. It is natural and interesting to ask the question whether or not a linear map is a derivation (Jordan derivation) if it is derivable (Jordan derivable) only at one given point. An element Z is called a linear full-derivable (Jordan full-derivable) point of $\mathcal{A}$ if every linear map from $\mathcal{A}$ into $\mathcal{M}$ that is derivable (Jordan derivable) at Z is in fact a derivation (Jordan derivation). It is surprising that there do exist full-derivable and Jordan full-derivable points for some algebras. For example, 0 and each invertible element (see [8, 14]) are full-derivable and Jordan full-derivable points of Banach algebras. More results can be found in [8–13] and references therein. Motivated by the study of derivable maps and Jordan derivations, it is more natural to introduce another Jordan derivable maps (see [15, 16] and references therein). In the following, we say that a linear map $\delta : \mathcal{A} \to \mathcal{M}$ is Jordan derivable at $Z \in \mathcal{A}$ if

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B) \text{ for any } A, B \in \mathcal{A} \text{ with } A \circ B = Z.$$

It was proved in [8] that a linear map from a prime Banach algebra into itself is Jordan derivable at the unit I if and only if it is a derivation. Similar results were obtained for a linear map from a triangular algebra into itself which is Jordan derivable at 0 and the unit I respectively. Notice that the equation $A \circ B = Z$ is more complicated than $AB = Z$, thus, so far there have been no articles on the study of linear maps Jordan derivable at a general Z. In this article, we investigate properties of linear maps Jordan derivable at a nontrivial idempotent P, and show that if $\mathcal{A} = \mathrm{Alg}\mathcal{N}$ is a Hilbert space nest algebra and $\mathcal{M} = \mathcal{B}(H)$, or $\mathcal{A} = \mathcal{M} = \mathcal{B}(X)$, then, a linear map $\delta : \mathcal{A} \to \mathcal{M}$ is Jordan derivable at a nontrivial projection $P \in \mathcal{N}$ or an arbitrary but fixed nontrivial idempotent $P \in \mathcal{B}(X)$ if and only if it is a derivation. Immediate consequences of our results generalize several results in the literature. It is worth mentioning that characterizing Jordan derivable maps needs deep knowledge from operator theory, but the former Jordan derivable maps can be characterized by pure algebraic method.

Let us fix some notation. Let X be a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. By X^* and $\mathcal{B}(X)$, we denote the topological dual space of X and the algebra of all linear bounded operators on X. A subspace lattice $\mathcal{L}$ of X is a collection of closed subspaces of X containing $\{0\}$ and X such that for every family $\{M_r\}$ of elements in $\mathcal{L}$, where both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$. For a subspace lattice $\mathcal{L}$, the associated subspace lattice algebra is $\mathrm{Alg}\mathcal{L} = \{A \in \mathcal{B}(X) : A(L) \subseteq L,\ \forall L \in \mathcal{L}\}$. A totally ordered subspace lattice is called a nest. If $\mathcal{L}$ is a nest, $\mathrm{Alg}\mathcal{L}$ is called a nest algebra. In the case that X is a Hilbert space, we change it to H. For Hilbert spaces, we disregard the distinction between a closed subspace and the orthogonal projection onto it.

2 Jordan Derivable Maps

In this section, we give an equivalent characterization of derivations on $\mathcal{B}(X)$ and a Hilbert space nest algebra by linear maps which are Jordan derivable at a nontrivial idempotent. First, we investigate properties of Jordan derivable maps.

Throughout this section, we assume that $\mathcal{A}$ is a unital algebra over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or