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DISTRIBUTIONS ON ALMOST CONTACT MANIFOLDS*

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Abstract It is known that any hypersurface in an almost complex space admits an almost contact manifold [11, 14]. In this article we show that a hyperplane in an almost contact manifold has an almost complex structure. Along with this result, we explain how to determine when an almost contact structure induces a contact structure, followed by examples of a manifold with a closed G_2-structure.

Key words Almost contact structure; contact structure; G_2-structure; foliation; integrability

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1 Introduction

Y. Tashiro [11] showed that an orientable hypersurface in an almost complex manifold has an almost contact structure by the use of tensor calculus. When we consider the converse, it is true in dimension 3 because every orientable 3-dimensional manifold is contact manifold. But it is not always true in all dimensions because S^4 as the hypersurface of contact manifold S^5 does not carry an almost complex structure. D. Blair and G. Ludden [3] proved that a hypersurface in an almost contact manifold is an almost complex manifold if the hypersuface is the normal direction to the characteristic vector field. In this article, we first show that the hyperplane distribution field of an almost contact manifold is an almost symplectic vector space, where the hyperplane is transverse to a vector field on an almost contact structure. Hence, an almost complex structure is naturally given to such hyperplane. Moreover, on a manifold of a closed G_2-structure, we show that there exists a hyperplane field with symplectic structure determined by the conformal vector fields. Another subject we address in this article is about the relationships between contact structures and almost contact structures of a smooth odd-dimensional manifold. Over the past decades, it has been investigated on the relationships between contact structures and almost contact structures. For instances, there are results of J. Martinet [9] and R. Lutz [8] on dimension 3, followed by a number of results on dimensions 5 by H. Geiges [7] and others [5], more recently one in a closed manifold [4]. However, there seems to be little knowledge about relationships in high dimensions from the geometric viewpoints. Hence, we bring our attention to a smooth seven-dimensional manifold and find some examples

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of results on it. As lots of orientable seven-dimensional manifolds admit G_2-structures on them, we mainly restrict our study to a manifold with a (closed) G_2-structure.

2 Preliminaries

We give brief introductions to the almost contact structure and G_2-structure in this section.

Definition 2.1 An almost contact structure on M of dimension $(2n+1)$ is a triple (J, α_ξ, ξ), where a tensor field J of type $(1,1)$, 1-form α_ξ, and nonvanishing vector field ξ which satisfy the following conditions:

- $\alpha_\xi(\xi) = 1$;
- $J^2 = -Id + \alpha_\xi \bigotimes \xi$.

From definition, we have the following relations between J, α_ξ, and ξ:

$$J(\xi) = 0, \quad \alpha_\xi(J) = 0. \tag{2.1}$$

The metric g is called a compatible metric if for any vector fields x, y, the metric g satisfies

$$g(Jx, Jy) = g(x, y) - \alpha_\xi(x)\alpha_\xi(y).$$

From this, we can see that a 1-form α_ξ is a metric dual 1-form of g because

$$g(Jx, \underbrace{J\xi}_{0}) = g(x, \xi) - \alpha_\xi(x) \underbrace{\alpha_\xi(\xi)}_{1}.$$

As $\alpha_\xi(J) = 0$ in (2.1), we have

$$\begin{aligned} g(x, Jy) &= g(Jx, J^2y) + \alpha_\xi(x) \underbrace{\alpha_\xi(Jy)}_{0} = g(Jx, -y + \alpha_\xi(y) \otimes \xi) \\ &= g(Jx, -y) + \alpha_\xi(y) \underbrace{g(Jx, \xi)}_{\alpha_\xi(Jx)=0}. \end{aligned}$$

This says that J is a skew-symmetric (1,1)-tensor with respect to the metric g. That is,

$$g(Jx, y) + g(x, Jy) = 0. \tag{2.2}$$

Every almost contact manifold admits a compatible metric g and the quadruple (J, α_ξ, ξ, g) is called an almost complex metric structure on M.

On a manifold M^{2n+1} with an almost contact metric structure (J, α_ξ, ξ, g), we can define a 2-form ω by $\omega(x, y) = g(x, Jy)$ and call ω a fundamental 2-form. Combining with (2.1) and (2.2), we obtain

$$\begin{aligned} \omega(Jx, Jy) &= g(Jx, J^2y) = g(Jx, -y + \alpha_\xi(y) \otimes \xi) \\ &= g(Jx, -y) + \alpha_\xi(y) \underbrace{g(Jx, \xi)}_{\alpha_\xi(Jx)=0} \\ &= g(x, Jy) = \omega(x, y). \end{aligned}$$

As J has rank $2n$, we have $\alpha_\xi \wedge \omega^n \neq 0$. When the fundamental 2-form ω is equal to $d\alpha_\xi$, an almost contact structure induces a contact structure because $\alpha_\xi \wedge \omega^n = \alpha_\xi \wedge (d\alpha_\xi)^n \neq 0$. By and large, on a contact manifold (M, α, R), where α is the contact 1-form with the Reeb vector field R, its hyperplane field H defined as the kernel of α carries a symplectic structure naturally given by $d\alpha$ and so admits compatible almost complex structure.