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# OUTER OPERATORS FOR THE NONCOMMUTATIVE SYMMETRIC HARDY SPACES ASSOCIATED WITH FINITE SUBDIAGONAL ALGEBRA*

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**Abstract** In this article, we extended main results on outer operators of [6] to the symmetric Hardy spaces, when associated subdiagonal algebra is finite.

**Key words** Subdiagonal algebra; noncommutative symmetric Hardy space; inner-outer operators; finite von Neumnann algebra

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## 1 Introduction

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state $\tau$. In [1], Arveson introduced the notion of finite, maximal, subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$, as non-commutative analogues of weak*- Dirichlet algebras. Subsequently, several authors studied the (non-commutative) $H^p$-spaces associated with such algebras(see [16]). Blecher and Labuschagne [5] studied outer operators in $H^p(\mathcal{A})$, $1 \leq p < \infty$ (for the case $0 < p < 1$, see [2]). In [6], the authors extend their generalized inner-outer factorization theorem in [5] and establish characterizations of outers that are valid even in the case of elements with zero determinant. Some of these results are proved in [19] for the case $0 < p < 1$.

In this article, we will extend main results on outer operators in [6] to the symmetric case, which can be considered as a complement to the work in [6] (also see [19]).

Section 1 contains some preliminary definitions. In Section 2, we extend the main results of [6] to the symmetric spaces case.

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## 2 Preliminaries

Throughout this article, we denote by $\mathcal{M}$ a finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal tracial state $\tau$. The closed densely defined linear operator $x$ in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators $u$ which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $x$ is affiliated with $\mathcal{M}$, then $x$ is said to be $\tau$-measurable if for every $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $e(H) \subseteq D(x)$ and $\tau(e^{\perp}) < \varepsilon$, where $e^{\perp} = 1-e$. The set of all $\tau$-measurable operators will be denoted by $L_0(\mathcal{M};\tau)$ or simply by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product to be the respective closure of the algebraic sum and product [15].

The measure topology $t_\tau$ in $L_0(\mathcal{M})$ is given by the system of neighborhoods of zero

$$V(\varepsilon,\delta) = \{x \in L_0(\mathcal{M}) : \|xe\|_\infty \le \delta \ \text{ for some pojection } e \in \mathcal{M} \text{ with } \tau(e^{\perp}) \le \varepsilon\},$$

$\varepsilon > 0, \delta > 0$.

For each $x$ on $\mathcal{H}$ affiliated with $\mathcal{M}$, all spectral projection $e_s^{\perp}(|x|) = \chi_{(s;\infty)}(|x|)$ corresponding to the interval $(s;\infty)$ belong to $\mathcal{M}$, and $x \in L_0(\mathcal{M})$ if and only if $\chi_{(s;\infty)}(|x|) < \infty$ for some $s \in \mathbb{R}$. Recall the definition of the decreasing rearrangement (or generalized singular numbers) of an operator $x \in L_0(\mathcal{M})$: for $t > 0$,

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \le t\}, t > 0,$$

where

$$\lambda_s(x) = \tau(e_s^{\perp}(|x|)); s > 0.$$

The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$. For more details on generalized singular value function of measurable operators, we refer to [10].

Now, we recall the definition of a symmetric operator space $E(\mathcal{M};\tau)$ build up with respect to a noncommutative measure space $(\mathcal{M},\tau)$ and a symmetric Banach function space (see [12]). Let $(\Omega,\Sigma,\mu)$ be a complete $\sigma$-finite measure space and $L_0(\Omega)$ be the space of all classes of $\mu$-measurable real-valued functions defined on $\Omega$. A quasi Banach space $E = (E, \|\cdot\|_E)$ is said to be a quasi Banach ideal space on $\Omega$ if $E$ is a linear subspace of $L_0(\Omega)$ and satisfies the so-called ideal property, which means that if $y \in E$, $x \in L_0(\Omega)$, and $|x(t)| = |y(t)|$ for $\mu$-almost all $t \in \Omega$, then $x \in E$ and $\|x\|_E = \|y\|_E$. By a symmetric quasi Banach space on $[0;1]$, we mean a quasi Banach lattice $E$ of measurable functions on $[0;1]$ satisfying the following properties:

(i) $E$ contains all simple functions;

(ii) If $x \in E$ and $y$ is measurable function such that $|y|$ is equidistributed with $|x|$, then $y \in E$ and $\|x\|_E = \|y\|_E$. For convience, we shall always assume that $E$ additionally satisfies

$$0 \le x_n \uparrow x, x_n, x \in E \Rightarrow \|x_n\|_E \uparrow \|x\|_E.$$

We say that $E$ has order continuous (quasi)-norm if for every net $(x_i)_{i\in I}$ in $E$ such that $x_i \downarrow 0$, we have $\|x_i\|_E \downarrow 0$. The space $E$ is called fully symmetric if, in addition, for $x \in L_0[0;1]$ and $y \in E$ with $x \prec\prec y$ implies that $x \in E$ and $\|x\|_E \le \|y\|_E$; here $x \prec\prec y$ as usual denotes the submajorization in the sense of Hardy-Littlewood-Polya: for all $t > 0$,

$$\int_0^t \mu_s(x)\mathrm{d}s \le \int_0^t \mu_s(y)\mathrm{d}s.$$