Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2017,**37B**(3):836–851
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

NONEXISTENCE AND SYMMETRY OF SOLUTIONS TO SOME FRACTIONAL LAPLACIAN EQUATIONS IN THE UPPER HALF SPACE*

Yanyan GUO (郭艳艳)
Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710129, China
E-mail: *yanyangcx@126.com*

Abstract In this article, we consider the fractional Laplacian equation

$$\begin{cases} (-\triangle)^{\alpha/2}u = K(x)f(u), & x \in \mathbb{R}^n_+, \\ u \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases}$$

where $0 < \alpha < 2, \mathbb{R}^n_+ := \{x = (x_1, x_2, \cdots, x_n) | x_n > 0\}$. When K is strictly decreasing with respect to $|x'|$, the symmetry of positive solutions is proved, where $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. When K is strictly increasing with respect to x_n or only depend on x_n, the nonexistence of positive solutions is obtained.

Key words Fractional Laplacian; method of moving planes; radial symmetry; nonexistence

2010 MR Subject Classification 35H20; 35J20

1 Introduction

This article is concerned with the semilinear fractional Laplacian equation

$$\begin{cases} (-\triangle)^{\alpha/2}u = K(x)f(u), & x \in \mathbb{R}^n_+, \\ u \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $0 < \alpha < 2, \mathbb{R}^n_+ := \{x = (x_1, x_2, \cdots, x_n) | x_n > 0\}$, $n \geq 2$.

The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator with the form

$$(-\triangle)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} \mathrm{d}z, \tag{1.2}$$

where $\alpha \in (0, 2)$. Let

$$L_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} \Big| \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} \mathrm{d}x < \infty \right\}.$$

It is easy to see that for $u \in L_\alpha \cap C^{1,1}_{\mathrm{loc}}$, the integral on the right hand side of (1.2) is well defined.

*Received April 29, 2016. This work is supported by the Fundamental Research Founds for the Central Universities (3102015ZY069) and the Natural Science Basic Research Plan in Shaanxi Province of China (2016M1008).

For Laplacian equations, the radially symmetric of positive solutions in $\mathbb{R}^n$ or a bounded domain was studied in [1–4] and the references therein, by using the "moving plane" technique. Later, Li, and Ni [5] obtained the symmetry results of positive solutions to (1.1) as $\alpha = 2, K(x) = K(|x|)$, by combining with the asymptotic behavior and the "moving plane" method. Guo [6] proved some non-existence results for positive weak solution to some semilinear elliptic system in the half-space by the Alexandrov-Serrin method of moving plane combined with integral inequality. In [7], Bianchi obtained the nonexistence of positive solutions to (1.1) as $\alpha = 2, K(x) = K(|x|)$. And then Damascelli and Gladiali [8] got new nonexistence results for a general class of semilinear elliptic problems in the half space with mixed (Dirichlet-Neumann) boundary conditions. For fractional Laplacian equation, Zhang [9] proved the symmetry of solutions to semilinear equations involving the fractional Laplacian. In [10], the monotonicity and nonexistence results for some fractional elliptic problems in the half space were obtained. Recently, Chen, Fang, and Yang [11] investigated the nonexistence of positive solutions to the fractional Laplacian on the upper half space

$$\begin{cases} (-\triangle)^{\alpha/2}u = u^p(x), & x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, & x \notin \mathbb{R}^n_+ \end{cases} \tag{1.3}$$

in the subcritical and critical case $1 < p \leq \frac{n+\alpha}{n-\alpha}$ using the method of moving planes in integral forms. In contrast, Chen, Li, and Li [12] worked directly on the non-local operators to develop a direct method of moving planes for fractional Laplacian. Using this method, they proved the radially symmetric and nonexistence of positive solutions to the fractional Laplacian equation

$$(-\triangle)^{\alpha/2}u = u^p, \quad x \in \mathbb{R}^n.$$

With different methods, many authors proved Liouville type results for nonnegative solutions of different kinds of fractional equations (see, for example, [13–16] and the references therein).

Therefore, the symmetry and nonexistence results of nonnegative solutions to (1.1) as $K(x) = 1$ were studied extensively. And so, it is natural to ask whether the same property of solutions still hold when $K(x)$ is not a constant.

In this article, we employ the method developed in [12] to prove the symmetry and nonexistence of positive solutions to (1.1) when K is not a constant in the upper half space.

Let $p > 1$ and $K \not\equiv 0$ throughout this article. The main results are stated in the following.

Theorem 1.1 Let $\beta \geq \frac{\alpha}{p-1}$. We denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and assume that $u = o(\frac{1}{|x|^\beta}) \in L_\alpha \cap C^{1,1}_{\rm loc}(\mathbb{R}^n_+)$ is a positive solution to

$$\begin{cases} (-\triangle)^{\alpha/2}u = K(|x'|, x_n)f(u), & x \in \mathbb{R}^n_+, \\ u \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases} \tag{1.4}$$

where K and f satisfy

(K_1) $K(|x'|, x_n) \geq 0$ is strictly decreasing with respect to $|x'|$ and is uniformly bounded for $x \in \mathbb{R}^n_+$;

(f_1) $f(u) > 0$ if $u > 0$, and $f(u)$ is Lipschitz continuous for $u \geq 0$;

(f_2) There exists a constant C such that

$$|f(u) - f(v)| \leqslant C|u - v|(|u|^{p-1} + |v|^{p-1})$$