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ON THE FOURIER-VILENKIN COEFFICIENTS*

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Abstract In this article, we prove the following statement that is true for both unbounded and bounded Vilenkin systems: for any $\varepsilon \in (0,1)$, there exists a measurable set $E \subset [0,1)$ of measure bigger than $1-\varepsilon$ such that for any function $f \in L^1[0,1)$, it is possible to find a function $g \in L^1[0,1)$ coinciding with f on E and the absolute values of non zero Fourier coefficients of g with respect to the Vilenkin system are monotonically decreasing.

Key words Vilenkin system; expansions; Fourier coefficients

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1 Introduction

Recall the definition of Vilenkin (multiplicative) systems of functions (see [1]). Consider the arbitrary sequence of natural numbers $P \equiv \{p_1, p_2, \cdots, p_k, \cdots\}$, where $p_j \geq 2$ for all $j \in \mathbb{N}$. We set

$$m_0 = 1,\ m_k = \prod_{j=1}^{k} p_j\ , \quad k \in \mathbb{N}. \tag{1.1}$$

It is not difficult to notice that for each point $x \in [0,1)$ and for any $n \in [m_{k-1}, m_k) \cap \mathbb{N},\ k \in \mathbb{N}$, there exist numbers $x_j, \alpha_j \in \{0, 1, \cdots p_j - 1\}$ such that

$$n = \sum_{j=1}^{k} \alpha_j m_{j-1} \text{ and } x = \sum_{j=1}^{\infty} \frac{x_j}{m_j},\ (P\text{-order expansions}). \tag{1.2}$$

Note that all points of type $\frac{l}{m_k}$ with $l, k \in \mathbb{N};\ 0 \leq l \leq m_k - 1$, have two different expansions: finite and infinite, and have only unique expansions we take only finite expansions for such points. As a result, we get the correspondences

$$n \to \{\alpha_1, \alpha_2, \cdots, \alpha_k\},\ x \to \{x_1, x_2, \cdots, x_k, \cdots\}. \tag{1.3}$$

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The Vilenkin system for sequence P is defined as follows:

$$W_0(x) \equiv 1;\ W_n(x) = \exp\left(2\pi \mathrm{i} \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right). \tag{1.4}$$

The expression (1.4) can be changed into the form

$$W_n(x) = \exp\left(2\pi \mathrm{i} \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right) = \prod_{j=1}^{k} \left(\exp\left(2\pi \mathrm{i} \frac{x_j}{p_j}\right)\right)^{\alpha_j}.$$

From (1.4), it follows that

$$W_{m_{j-1}}(x) = \exp\left(2\pi \mathrm{i} \frac{x_j}{p_j}\right),$$

and for the n-th function, we obtain the expression

$$W_n(x) = \prod_{j=1}^{k} (W_{m_{j-1}}(x))^{\alpha_j}.$$

Notice that

$$\int_0^1 W_n(t)\overline{W}_k(t)\mathrm{d}t = \begin{cases} 1, & \text{if } k = n; \\ 0, & \text{if } k \neq n, \end{cases} \quad \text{where } \overline{W}_k(t) \text{ is the complex conjugate of } W_k(t).$$

It is obvious that systems, corresponding to different sequences $\{p_k\}$, differ from each other (if $P \equiv \{2, 2, \cdots, 2, \cdots\}$, Vilenkin system coincides with the Walsh system (see [1])). If $\sup\{p_k\} = \infty$ ($\sup\{p_k\} < \infty$), the system $\{W_n(x)\}$ is said to be unbounded (accordingly bounded).

The theory of such systems have been introduced by N. Ja. Vilenkin in 1946 (see [2, 3]). Then, there are interesting results for Vilenkin system (see [4–7]).

In 1957, C. Watari [8] proved that the bounded Vilenkin system is basis in L^r when $r > 1$. Then, in 1976, W.S. Young [9] for arbitrary sequence $\{p_k\}$ (that is, both for bounded and unbounded Vilenkin systems) established the basicity of Vilenkin system in L^r when $r > 1$. For any function $f \in L^1[0, 1)$ and for all $n \in \mathbb{N}$ and $y \in (0, \infty)$, he also proved the inequality

$$\mathrm{mes}\{x : |S_n(x, f)| > y\} \leq \frac{C\|f\|_{L^1[0,1)}}{y}, \text{ where } C - \text{ is constant.}$$

Note that the following problem remains open: Is the Fourier series of function from $L^2[0, 1)$ with respect to the unbounded Vilenkin systems convergent almost everywhere or not?

Note also that in [10], P. Billard established that this problem has a positive answer for the Walsh system.

Let $f(x)$ be a real valued function from $L^r[0, 1),\ r \geq 1$ and $c_n(f)$ be the Fourier-Vilenkin coefficients of function f, that is,

$$c_n(f) = \int_0^1 f(x)\overline{W}_n(x)\mathrm{d}x. \tag{1.5}$$

Let $\mathrm{spec}(f)$ be the spectrum of $f(x)$, that is, the set of integers k for which $c_k(f) \neq 0$.

In this article, we prove the following:

Theorem 1.1 Let $\{W_k(x)\}_{k=0}^{\infty}$ — be either unbounded or bounded Vilenkin system. Then, for each $0 < \epsilon < 1$, there exists a measurable set $E \subset [0, 1)$ of measure $|E| > 1 - \varepsilon$ such that for any function $f \in L^1[0, 1)$, there exists a function $g \in L^1[0, 1)$ such that $f(x) = g(x)$ if $x \in E$ and the elements of sequence $\{|c_k(g)|\ ,\ k \in \mathrm{spec}(g)\}$ are monotonically decreasing.