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UNIFORM QUASI-DIFFERENTIABILITY OF SEMIGROUP TO NONLINEAR REACTION-DIFFUSION EQUATIONS WITH SUPERCRITICAL EXPONENT*

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Abstract A new approach is established to show that the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by a reaction-diffusion equation with supercritical exponent is uniformly quasi-differentiable in $L^q(\Omega)$ $(2\leqslant q<\infty)$ with respect to the initial value. As an application, this proves the upper-bound of fractal dimension for its global attractor in the corresponding space.

Key words Uniform quasi-differentiability; semigroup; reaction-diffusion equation

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1 Introduction

In this article, we are mainly concerned with the uniform quasi-differentiability with respect to the initial value and the estimate of fractal dimension of global attractor for the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by the following equation:

$$\begin{cases} u_t-\Delta u+\mu|u|^{p-1}u+\lambda f(x,u)=g(x) & \text{in } \Omega\times\mathbb{R}^+,\\ u=0 & \text{on } \partial\Omega\times\mathbb{R}^+,\\ u(x,0)=u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^n (n\geqslant 1)$; the parameters $\mu\in\mathbb{R}^+,\lambda\in\mathbb{R}$; the nonlinear function $f\in C^1$ satisfies the following condition

$$c_1|u|^{s-2}u-c_0\leqslant f'(x,u)\leqslant c_2|u|^{s-2}u+c_0,\quad 1\leqslant s<p; \tag{1.2}$$

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and the dominant growth exponent $p(>2)$ in the nonlinear term "$|u|^{p-1}u$" is supercritical, namely, $p > 2^* = \frac{2n}{n-2}$ and the external force $g(x) \in L^2(\Omega)$ (note that it is required in [1] that $g(x) \in L^\infty(\Omega)$).

As we known that if we want to describe the Lyapunov exponents and Lyapunov numbers (see [2] for more details), or to prove the existence of exponential attractor (see [3, 4]), or to show the existence of inertial manifold (including the structure of a mapping in the neighborhood of a fixed point), we usually need to consider the differentiability of semigroup on some absorbing set or some fixed point; see [5–11] and the references therein. The differentiability of semigroup may be proved in especially simple way in the case when the space E consists of such smooth functions that all the derivatives with respect to spaceous variables which are included in the principle differential operator (for instance "$-\Delta$" in our case) are bounded. However, to prove differentiability of semigroup in spaces like $L^2(\Omega)$ or of $H_0^1(\Omega)$, one usually need to impose further restrictions on the smoothness and growth of nonlinear function; for example, see [2, 12].

It is well known that, to prove the differentiability of semigroup generated by parabolic equation, one used to add the following standard assumption for the nonlinear term $F(u)$ (which was called Hölder condition) (for example, see Theorem 1.3 in [12], or Theorem 5.9 in [13]):

$$|F'(u) - F'(v)| \leqslant C|u-v|^\beta \tag{1.3}$$

with $0 < \beta \leqslant 1$, which implies that the nonlinear term is of class $C^{1+\beta}$(see [12], P345).

Consequently, there is a natural question that whether or not the semigroup generated by equation (1.1) with supercritical exponent is also differentiable(in some sense) when the above standard assumption is not valid.

Here, we show that the semigroup generated by equation (1.1) with supercritical exponent is uniformly quasi-differentiable in space $L^2(\Omega)$(moreover, $L^q(\Omega)(q \geqslant 2)$) when the initial value belongs to some appropriate absorbing set. It is worth noting that for our results even the Hölder condition (1.3) may not hold, because there is an equilibrium point which may only belong to $L^{2p}(\Omega)$ (consequently it is impossible to obtain the constant C in (1.3)) (see Proposition (1.1) below). And it shall be mentioned that we extends the assertion proved in [14] that the above semigroup is only norm-to-weak.

We present firstly the following result concerning the initial value (that is a absorbing set) of semigroup.

Proposition 1.1 Assume that the functions f, g satisfy the above assumptions, then there exists a bounded absorbing set $\mathbb{D} = u^* + \widetilde{\mathbb{D}}(\subset L^{2p}(\Omega))$ for semigroup $\{S(t)\}_{t\geqslant 0}$ generated by equation (1.1). Here, $u^*(\in L^{2p}(\Omega)$ merely) is the ground state of the associated elliptic equation (3.1) and the regular set $\widetilde{\mathbb{D}}(\subset L^q(\Omega))$ (for any $2 \leqslant q < \infty$) is a bounded absorbing set of the following equation (3.3).

In order to prove this proposition, motivated by [15], we similarly make a shift: accordingly set $S_1(t)(u_0 - u^*) = S(t)u_0 - u^*$, where u^* is the ground state of corresponding elliptic equation (3.1) and the semigroup $\{S_1(t)\}_{t\geqslant 0}$ satisfies the shifting equation (3.3).

Next, considering the relationship of differentiability between the correlative semigroup $\{S(t)\}_{t\geqslant 0}$ and $\{S_1(t)\}_{t\geqslant 0}$ generated by equations (1.1) and (3.3) respectively (see Lemma 3.4),