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ALMOST CONSERVATION LAWS AND GLOBAL ROUGH SOLUTIONS OF THE DEFOCUSING NONLINEAR WAVE EQUATION ON $\mathbb{R}^2$*

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Abstract In this article, we investigate the initial value problem(IVP) associated with the defocusing nonlinear wave equation on $\mathbb{R}^2$ as follows:

$$\begin{cases} \partial_{tt}u - \triangle u = -u^3, \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), \end{cases}$$

where the initial data $(u_0, u_1) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$. It is shown that the IVP is global well-posedness in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ for any $1 > s > \frac{2}{5}$. The proof relies upon the almost conserved quantity in using multilinear correction term. The main difficulty is to control the growth of the variation of the almost conserved quantity. Finally, we utilize linear-nonlinear decomposition benefited from the ideas of Roy [1].

Key words Defocusing nonlinear wave equation; global well-posedness; I-method; linear-nonlinear decomposition; below energy space

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1 Introduction

In this article, we consider the IVP with the defocusing nonlinear wave equation on $\mathbb{R}^2$ as follows:

$$\begin{cases} \partial_{tt}u - \triangle u = -u^3, \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

where u is an unknown real function defined on $[0;T] \times \mathbb{R}^2$, and the initial data $(u_0, u_1) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ with $s < 1$. Here, H^s is the usual inhomogeneous Sobolev space, that is, H^s is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|f\|_{H^s(\mathbb{R}^2)} := \|(1+D^s)f\|_{L^2(\mathbb{R}^2)},$$

where D is the operator defined by

$$\widehat{Df}(\xi) := |\xi|\hat{f}(\xi)$$

and $\hat{f}$ denotes the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}^2} f(x)e^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x.$$

Here, $H^s \times H^{s-1}$ is the product space of H^s and H^{s-1} endowed with the standard norm $\|(f,g)\|_{H^s\times H^{s-1}} := \|f\|_{H^s} + \|g\|_{H^{s-1}}$. Especially, $\dot{H}^s$ is the usual homogeneous Sobolev space, that is, the completion of Schwartz functions $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \|D^s f\|_{L^2(\mathbb{R}^2)}.$$

The solution u of problem (1.1) has the following energy conservation law

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u(t,x)|^2 + u_t^2(t,x))\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} u^4(t,x))\mathrm{d}x = E(u(0)). \tag{1.2}$$

In the case of 3-D, problem (1.1) was shown by Lindbald and Sogge [2] to be locally well-posed in $H^s(\mathbb{R}^3)$, $s > \frac{1}{2}$. More precisely, for any $(u_0,u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ with $s > \frac{1}{2}$, there exist a time $T = T(\|(u_0,u_1)_{H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)}\|)$ and a unique solution-pair (u, u_t) of (1.1) in a certain Banach space $X \subset C([0,T]; H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3))$. Furthermore, the solution map is continuous from $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ to $C([0,T]; H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3))$.

Now, we pay attention to the global well-posedness theory of (1.1). If the lifetime T of the solution can be taken arbitrarily large, we say that problem (1.1) is globally well-posed in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$. This topic was widely investigated. First of all, we immediately obtain the global well-posedness in the energy space $H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ (that is, $s = 1$) combining the energy conservation law with the local well-posedness theory. However, in this article, we are interested in investigating the global well-posedness of (1.1) for the initial data whose regularity is below the energy, that is, $s < 1$. For the initial data below energy space $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$, there are several works related 3-D case. As for 3-D case, Kenig, Ponce, and Vega [3] proved that (1.1) is globally well-posed in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for any $1 > s > \frac{3}{4}$ using the Fourier truncation method benefited from the idea of Bourgain [4]. Also, Gallagher and Planchon [5] proved the same result by different method. Later on, Bahouri and Chemin [6] proved the global well-posedness for $s = \frac{3}{4}$ using a nonlinear interpolation method and logarithmic estimates benefited from the idea of Klainerman and Tataru [7]. More recently, using the