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THE LARGEST EIGENVALUE DISTRIBUTION OF THE LAGUERRE UNITARY ENSEMBLE*

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Abstract We study the probability that all eigenvalues of the Laguerre unitary ensemble of n by n matrices are in $(0, t)$, that is, the largest eigenvalue distribution. Associated with this probability, in the ladder operator approach for orthogonal polynomials, there are recurrence coefficients, namely, $\alpha_n(t)$ and $\beta_n(t)$, as well as three auxiliary quantities, denoted by $r_n(t)$, $R_n(t)$, and $\sigma_n(t)$. We establish the second order differential equations for both $\beta_n(t)$ and $r_n(t)$. By investigating the soft edge scaling limit when $\alpha = O(n)$ as $n \to \infty$ or α is finite, we derive a P_{II}, the σ-form, and the asymptotic solution of the probability. In addition, we develop differential equations for orthogonal polynomials $P_n(z)$ corresponding to the largest eigenvalue distribution of LUE and GUE with n finite or large. For large n, asymptotic formulas are given near the singular points of the ODE. Moreover, we are able to deduce a particular case of Chazy's equation for $\varrho(t) = \Xi'(t)$ with $\Xi(t)$ satisfying the σ-form of P_{IV} or P_V.

Key words Orthogonal polynomials; Painlevé equations; differential equations

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1 Introduction

A unitary ensemble is well defined for Hermitian matrices $M = (M_{ij})_{n\times n}$ with probability density

$$p(M)dM \propto e^{-\mathrm{tr}\, v(M)}\mathrm{vol}(dM), \quad \mathrm{vol}(dM) = \prod_{i=1}^{n} dM_{ii} \prod_{1\le j<k\le n} d(\mathrm{Re}M_{jk})d(\mathrm{Im}M_{jk}). \tag{1.1}$$

Here, $v(M)$ is a matrix function [1] defined via Jordan canonical form and vol(dM) is called the volume element [2]. The joint probability density function of the eigenvalues $\{x_j\}_{j=1}^n$ of this unitary ensemble is given in [3] by

$$\frac{1}{D_n(a,b)}\,\frac{1}{n!}\prod_{1\le j<k\le n}|x_k - x_j|^2\prod_{j=1}^{n} w(x_j), \tag{1.2a}$$

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where $D_n(a,b)$ is the normalization constant which reads

$$D_n(a,b) = \frac{1}{n!}\int_{[a,b]^n} \prod_{1\le j<k\le n} |x_k - x_j|^2 \prod_{j=1}^{n} w(x_j)\mathrm{d}x_j, \tag{1.2b}$$

and $w(x) = e^{-v(x)}$ is a positive weight function supported on $[a,b]$ with finite moments

$$\mu_k := \int_a^b x^k w(x)\mathrm{d}x, \qquad k = 0,1,2,\cdots.$$

It is shown, in [3], that $D_n(a,b)$ can be evaluated as the determinant of the Hankel (or moment) matrix, that is,

$$D_n(a,b) = \det\left(\mu_{i+j}\right)_{i,j=0}^{n-1}.$$

A unitary ensemble is called the Laguerre unitary ensemble (LUE) if in (1.1),

$$v(x) = x - \alpha \ln x,$$

or, what amounts to the same thing, in (1.2),

$$w(x) = x^{\alpha}e^{-x}, \qquad x \in [0,\infty), \quad \alpha > 0.$$

A special case of LUE is $M = XX^*$ and $\alpha = p - n$, where $X = X_1 + \mathrm{i}X_2$ is an $n \times p$ $(n \le p)$ random matrix with each element of X_1 and X_2 chosen independently as a Gaussian random variable; see [4–7].

Denote by $\mathbb{P}(n,t)$ the probability that the largest eigenvalue in LUE is not larger than t, then

$$\mathbb{P}(n,t) = \frac{D_n(t)}{D_n(0,\infty)},$$

where $D_n(t) := D_n(0,t)$. Tracy and Widom [8] obtained the Jimbo-Miwa-Okamoto (J-M-O) σ-form [9, 10] of P_V for

$$\sigma_n(t) := t\frac{\mathrm{d}}{\mathrm{d}t}\ln \mathbb{P}(n,t)$$

using the Fredholm determinant. Basor and Chen [11] derived the same σ-form by studying the Hankel determinant $D_n(t)$ with the help of the ladder operators related to orthogonal polynomials. In their work, another four quantities associated with $\mathbb{P}(n,t)$ are considered, that is, $\alpha_n(t)$, $\beta_n(t)$, $r_n(t)$, and $R_n(t)$, and the relationships between them are established. In addition, a P_V is derived for $R_n(t)$ (or $\alpha_n(t)$). By these results, in this article we obtain the second order differential equation for $\beta_n(t)$ as well as $r_n(t)$.

The soft edge scaling limit of the smallest eigenvlue distribution on (t,∞) in LUE with $\alpha = \mu n = O(n)$ and $t = \left(\sqrt{\mu+1}-1\right)^2 n - \frac{\left(\sqrt{\mu+1}-1\right)^{4/3}}{(\mu+1)^{1/6}} n^{1/3}s$ is analyzed in [12]. Concerning the largest eigenvalue distribution, we show that for $\alpha = O(n)$ or finite, and

$$t = c_1 n + c_2 n^{1/3}s, \quad \sigma(s) := \frac{c_2}{c_1}\lim_{n\to\infty} n^{-2/3}\sigma_n(t)$$

where

$$c_1 = \left(\sqrt{\mu+1}+1\right)^2, \quad c_2 = \frac{\left(\sqrt{\mu+1}+1\right)^{4/3}}{(\mu+1)^{1/6}}, \quad \mu = \begin{cases} \frac{\alpha}{n}, & \alpha = O(n) \\ 0, & \alpha \text{ is finite} \end{cases},$$

the σ-form of P_V satisfied by $\sigma_n(t)$ is reduced down to the σ-form of P_{II}, which agrees with the result of [12]. The P_V, the ODEs for $\beta_n(t)$ and $r_n(t)$, can likewise be reduced to a P_{II}.