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A VARIATIONAL PROBLEM ARISING IN REGISTRATION OF DIFFUSION TENSOR IMAGES*

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Abstract The existence of a global minimizer for a variational problem arising in registration of diffusion tensor images is proved, which ensures that there is a regular spatial transformation for the registration of diffusion tensor images.

Key words Variational problem; image matching; registration; diffusion tensor image

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with Liptschitz boundary $\partial\Omega$, and let T and D be two functions such that

$$T : \Omega \to \mathbb{R}^d \quad \text{and} \quad D : \Omega \to \mathbb{R}^d \quad (d \geq 1). \tag{1.1}$$

By talking about the image registration, T and D are viewed as two images in a spatial domain Ω, one is called the floating image (for example, T) and the other is the target image (for example, D). The registration of two images is to find a smooth and locally non-degenerate spatial transformation $h : \Omega \to \Omega$ such that the composition of T and h, that is $T \circ h(x) = T(h(x))$, approaches D as possible in some sense, for example, find a h such that the sum of squared errors in a suitable space reaches its minimum. Note that the dimension d in (1.1) of the space for the ranges of T and D varies with imaging modalities, for examples, images acquired by Magnetic Resonance Imaging(MRI) or Computed Tomography(CT) are scalar valued with $d = 1$, RGB images are vector valued with $d = 3$, and the Diffusion Tensor Imaging(DTI) is matrix valued with $d = 9$ (or $d = 6$ if the matrix at each voxel of DTI is 3×3 symmetric and positive).

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In order to get a spatial transformation $h:\Omega\to\Omega$ with higher regularity, Dupuis, Grenander, and Miller [5] improved the variational model proposed by Amit [2] for scalar image registration ($d=1$), and they considered the following variational problem in a suitable space for v:

$$\hat{v}=\arg\min_{v}\int_0^{\tau}\|Lv(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t+\|T\circ h(\cdot)-D(\cdot)\|^2_{L^2(\Omega)},\tag{1.2}$$

where $\tau>0$ is the time duration, "$\circ$" is the composition of two functions, and $L:[H_0^3(\Omega)]^3\to[L^2(\Omega)]^3$ is a differential operator satisfying

$$\|Lv(\cdot,t)\|^2_{L^2(\Omega)}=\sum_{i=1}^{3}\int_\Omega|(Lv)_i(x,t)|^2\mathrm{d}x\geq c\|v(\cdot,t)\|^2_{[H_0^3(\Omega)]^3},\tag{1.3}$$

for some $c>0$ and for all $v\in[H_0^3(\Omega)]^3\triangleq H_0^3(\Omega)\times H_0^3(\Omega)\times H_0^3(\Omega)$ with $H_0^3(\Omega)=W_0^{3,2}(\Omega)$. Clearly, $L=\sum\limits_{i+j+k=3}\frac{\partial^3}{\partial x_1^i\partial x_2^j\partial x_3^k}$ satisfies (1.3). Furthermore, v and h are constrained by the following equation:

$$\frac{\mathrm{d}\eta(s;t,x)}{\mathrm{d}s}\triangleq\dot{\eta}(s;t,x)=v(\eta(s;t,x),s),\ \ \eta(t;t,x)=x,\ \ h(x)=\eta(0;\tau,x),\ \ 0\leq s,t\leq\tau,\tag{1.4}$$

here $\eta(s;t,x)$ means that a particle placed at $y=\eta(s;t,x)$ at time s is transformed to a point x at time t under the forcing term $v(x,t)$. In [5], the authors gave a rigid mathematical proof on the existence of global minimizer, which validates the applications of model (1.2) in numerical simulations of scalar image registration. Based on [5], the large derivation principle(LDP) of the constraint equation (1.4) are concerned in [4]. However, the model (1.2) does not work for registration of DTI images because each voxel of DTI image contains a 3×3 symmetric positive definite real matrix (that is, diffusion tensor) and the orientation of diffusion tensors must be considered in making spatial transformation, which is much more complicated than that of scalar images. For this reason, there are two tensor reorientation strategies, the finite strain(FS) and the preservation principle direction(PPD), have been proposed by Alexander[1], which are widely used for analyzing DTI data. Note that, for DTI images, both T and D are maps from Ω to the set of 3×3 Symmetric Positive Definite real matrixes ($SPD(3)$ in short), that is,

$$T,D:\Omega\to SPD(3)\subset\mathbb{R}^6.\tag{1.5}$$

In order to extend the variational model (1.2) to the case of registration of DTI images, based on FS reorientation strategy, Li et al [8] introduced a new transformation operation "$\diamond$" which is used to replace the usual composition operation "$\circ$" in (1.2). The operation "$\diamond$" is given by

$$T\diamond h(x)=R\left[T\circ h(x)\right]R^T\ \text{ with }\ R=J_x^T\left(J_xJ_x^T\right)^{-\frac{1}{2}}\text{ and }\ J_x=\nabla_x h^{-1}(x),\tag{1.6}$$

and the computation of $\left(J_xJ_x^T\right)^{-\frac{1}{2}}$ is given in Appendix.

With the above definition and notations, the variational model proposed in [8] for the registration of DTI images can be formulated by

$$\hat{v}=\arg\min_{v\in\mathcal{F}}H(v),\tag{1.7}$$

where $H(v)=\int_0^\tau\|Lv(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t+\|T\diamond h(\cdot)-D(\cdot)\|^2_{L^2(\Omega)}$, τ and L are the same as that in (1.2), $h(x)$ is given by (1.4), and $\mathcal{F}$ is defined by (1.10).