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# A NECESSARY AND A SUFFICIENT CONDITION FOR THE EXISTENCE OF THE POSITIVE RADIAL SOLUTIONS TO HESSIAN EQUATIONS AND SYSTEMS WITH WEIGHTS*

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**Abstract** In this article, we consider the existence of positive radial solutions for Hessian equations and systems with weights and we give a necessary condition as well as a sufficient condition for a positive radial solution to be large. The method of proving theorems is essentially based on a successive approximation technique. Our results complete and improve a work published recently by Zhang and Zhou (existence of entire positive $k$-convex radial solutions to Hessian equations and systems with weights. Applied Mathematics Letters, Volume 50, December 2015, Pages 48–55).

**Key words** existence; Keller-Osserman condition; $k$-Hessian equation and system

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## 1 Introduction

Let $D^2u$ be the Hessian matrix of a $C^2$ (i.e., a twice continuously differentiable) function $u$ defined over $\mathbb{R}^N$ and

$$\lambda\left(D^2u\right)=(\lambda_1,\cdots,\lambda_N)$$

the vector of eigenvalues of $D^2u$.

For $k=1,2,\cdots,N$ define the $k$-Hessian operator as follows

$$S_k\left(\lambda\left(D^2u\right)\right)=\sum_{1\le i_1<\cdots<i_k\le N}\lambda_{i_1}\cdots\lambda_{i_k},$$

i.e., it is the $k^{th}$ elementary symmetric polynomial of the Hessian matrix of $u$. In other words, $S_k\left(\lambda\left(D^2u\right)\right)$ is the sum of all $k\times k$ principal minors of the Hessian matrix $D^2u$ and so is a second order differential operator, which may also be called the $k$-trace of $D^2u$. Especially, it is easy to see that the $N$-Hessian is the Monge-Ampére operator and that the 1-Hessian is the well known classical Laplace operator. Hence, the $k$-Hessian operators form a discrete collection of partial differential operators which includes both the Laplace and the Monge-Ampére operator.

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In this paper, we study the existence of radial solutions for the following Hessian equation

$$S_k^{1/k}\left(\lambda\left(D^2u\right)\right)=p\left(|x|\right)h\left(u\right),\ x\in\mathbb{R}^N\ \ (N>2k) \tag{1.1}$$

and system

$$\begin{cases} S_k^{1/k}\left(\lambda\left(D^2u\right)\right)=p\left(|x|\right)f\left(u,v\right),\ x\in\mathbb{R}^N\ , \\ S_k^{1/k}\left(\lambda\left(D^2v\right)\right)=q\left(|x|\right)g\left(u,v\right),\ x\in\mathbb{R}^N \end{cases}\ (\ N>2k). \tag{1.2}$$

Setting $r=|x|$ we assume that the functions $p$, $q:[0,\infty)\to(0,\infty)$, $h:[0,\infty)\to[0,\infty)$ and $f$, $g:[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfy some of the conditions:

(P1) $p$, $q$ are continuous;

(P2) $r^{\frac{(k+1)N}{k}-2}p^k\left(r\right)$ is nondecreasing;

(P3) $r^{\frac{(k+1)N}{k}-2}\left[p^k\left(r\right)+q^k\left(r\right)\right]$ is nondecreasing;

(C1) $h$ is monotone non-decreasing, $h(0)=0$ and $h\left(s\right)>0$ for all $s>0$;

(C2) $f$, $g$ are monotone non-decreasing in each variable, $f(0,0)=g\left(0,0\right)=0$ and $f(s,t)>0$, $g\left(s,t\right)>0$ for all $s,t>0$;

(C3) $\int_1^\infty\frac{1}{\sqrt[k+1]{(k+1)H(t)}}\mathrm{d}t=\infty$ for $H(t)=\int_0^t h^k(z)\mathrm{d}z$;

(C4) $\int_1^\infty\frac{1}{\sqrt[k+1]{(k+1)F(t)}}\mathrm{d}t=\infty$ for $F(t)=\int_0^t\left(f^k(z,z)+g^k(z,z)\right)\mathrm{d}z$.

The properties of the $k$-Hessian operator were well discussed in a numerous papers written by Ivochkina (see [8–11] and others). Moreover, this operator appears as an object of investigation by many remarkable geometers. For example, Viaclovsky (see [21, 22]) observed that the $k$-Hessian operator is an important class of fully nonlinear operators which is closely related to a geometric problem of type (1.1) (see Bao-Ji-Li [2] for a more detailed discussion). Also, equation (1.1) arises via the study of the quasilinear parabolic problem (see for example the introduction of Moll-Petitta [17]).

In the present work we will limit ourselves to the development of mathematical theory for (1.1) and (1.2). The main difficulty in investigating these problems, such as (1.1) or (1.2), in which the $k$-Hessian operator appears is related to the fact that their properties change depending on the subset of $C^2$ from which the solution is taken.

Our main objective here is to find functions in $C^2$ that are strictly $k$-convex and verify problems (1.1), (1.2), where by strictly $k$-convex function $u$ we mean that all eigenvalues $\lambda_1,\cdots,\lambda_N$ of the symmetric matrix $D^2u$ are in the so called Gårdding open cone $\Gamma_k$ which is defined by

$$\Gamma_k\left(N\right)=\left\{\lambda\in\mathbb{R}^N\left|S_1\left(\lambda\right)>0,\cdots,S_k\left(\lambda\right)>0\right.\right\}.$$

In the next we adopt the notation from Li [13] for the space of all admissible functions

$$\Phi^k\left(\mathbb{R}^N\right):=\left\{u\in C^2\left(\mathbb{R}^N\right)\left|\lambda\in\Gamma_k\left(N\right)\ \text{ for all } x\in\mathbb{R}^N\right.\right\}.$$

Regarding our work, there were some recent papers resolving the existence for blow-up solutions of (1.1) and (1.2). Here we wish to mention the works of Bao-Ji-Li [2], Hamydy [6], Jacobsen [12], Bao-Ji [13], Lazer-McKenna [15, (the case $k=N$)], Li-Zhang-Zhang [16], Salani [19], Singh [20],Yang [23], Zhang-Liu-Wu [24], Zhang-Zhou [25], Zhang-Shi-Xue [26] and Zhang [27] which will be useful in our proofs.

Motivated by the recent work of Zhang-Zhou [25] we are interested in proving the following theorems.