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SOME PROPERTIES FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS DEFINED BY DIFFERENTIAL INEQUALITIES*

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Abstract Let $\mathcal{A}$ be the space of functions analytic in the unit disk $D = \{z : |z| < 1\}$. Let $\mathcal{U}$ denote the set of all functions $f \in \mathcal{A}$ satisfying the conditions $f(0) = f'(0) - 1 = 0$ and

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1 \quad (|z| < 1).$$

Also, let Ω denote the set of all functions $f \in \mathcal{A}$ satisfying the conditions $f(0) = f'(0) - 1 = 0$ and

$$|zf'(z) - f(z)| < \frac{1}{2} \quad (|z| < 1).$$

In this article, we discuss the properties of $\mathcal{U}$ and Ω.

Key words univalent function; Starlike function; Hadamard product; extreme point; support point

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1 Introduction

Let $\mathcal{A}$ be the space of functions analytic in the unit disk $D = \{z : |z| < 1\}$. Endowed with the topology of uniform convergence on compact subsets of the unit disk, $\mathcal{A}$ is a locally convex topological vector space. Let $\mathcal{B}$ be the subset of $\mathcal{A}$ which consists of $\phi \in \mathcal{A}$ with $|\phi(z)| \le 1$ $(|z| < 1)$. Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$. A function f is called a support point of $\mathcal{F}$ if $f \in \mathcal{F}$ and there is a continuous linear functional J on $\mathcal{A}$ such that ReJ is non-constant on $\mathcal{F}$ and

$$\mathrm{Re}J(f) = \max\{\mathrm{Re}J(g) : g \in \mathcal{F}\}.$$

The set of all support points of $\mathcal{F}$ is denoted by supp$\mathcal{F}$. An element $f \in \mathcal{F}$ is called an extreme point of $\mathcal{F}$ if it is not a proper convex combination of any two distinct points in $\mathcal{F}$. The set of extreme points of $\mathcal{F}$ is denoted by E$\mathcal{F}$. In recent decades the extreme points and support points of many classes of analytic functions were studied (see [1–12]).

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Let S be the subset of $\mathcal{A}$ consisting of functions f that are univalent in D and satisfy $f(0) = f'(0) - 1 = 0$. A function $f \in S$ is called starlike if $f(D)$ is starlike with respect to the origin. The class of all starlike functions is denoted by S^*. A function $f \in S^*$ if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \quad (|z| < 1).$$

A function $f \in S$ is called convex if $f(D)$ is a convex set. The class of all convex functions is denoted by K. A function $f \in K$ if and only if

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0 \quad (|z| < 1).$$

In recent years, many scholars were still interested in studying the properties of various subclasses of univalent functions (see [13–20]).

Let $\mathcal{U}$ denote the set of all functions $f \in \mathcal{A}$ satisfying the conditions $f(0) = f'(0) - 1 = 0$ and

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1 \quad (z \in D). \tag{1.1}$$

Let $h(z) = \frac{z}{f(z)} - 1$. It is easy to show that $f \in \mathcal{U}$ if and only if

$$zh'(z) - h(z) = z^2\phi(z), \tag{1.2}$$

where $\phi \in \mathcal{B}$, that is, ϕ is analytic in D and $|\phi(z)| \le 1$ for all $z \in D$. It is well known that the functions in $\mathcal{U}$ are univalent [15]. Up to now, the class $\mathcal{U}$ were studied in detail for many years (see [16–20]). In the second part of this paper, we further study the class $\mathcal{U}$, and meanwhile, we want to study the univalence of the function $h(z)$ which is a solution of the differential equation (1.2). For this purpose, we define a new class Ω which consists of functions $f \in \mathcal{A}$ satisfying the conditions $f(0) = f'(0) - 1 = 0$ and

$$|zf'(z) - f(z)| < \frac{1}{2} \quad (z \in D). \tag{1.3}$$

It is clear that inequality (1.3) is equivalent to the equation

$$zf'(z) - f(z) = \frac{1}{2}z^2\phi(z), \tag{1.4}$$

where $\phi \in \mathcal{B}$. In the third part of this article, we focus on the properties of the class Ω such as univalence, radius of convexity, Hadamard product, extreme and support points. At last, we put forward a question to end this article.

Now, we state our main results.

2 The Property of the Class $\mathcal{U}$

Theorem 2.1 If $f, g \in \mathcal{U}, 0 \le t \le 1$, then

$$\frac{fg}{(1-t)f + tg} \in \mathcal{U}.$$

Proof Since $f, g \in \mathcal{U}$, we conclude from (1.2) that

$$z\left[\frac{z}{f(z)} - 1\right]' - \left[\frac{z}{f(z)} - 1\right] = z^2\varphi(z) \tag{2.1}$$