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THE POINTWISE ESTIMATES OF SOLUTIONS FOR A NONLINEAR CONVECTION DIFFUSION REACTION EQUATION*

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Abstract This paper studies the time asymptotic behavior of solutions for a nonlinear convection diffusion reaction equation in one dimension. First, the pointwise estimates of solutions are obtained, furthermore, we obtain the optimal $L^p, 1 \leqslant p \leqslant +\infty$, convergence rate of solutions for small initial data. Then we establish the local existence of solutions, the blow up criterion and the sufficient condition to ensure the nonnegativity of solutions for large initial data. Our approach is based on the detailed analysis of the Green function of the linearized equation and some energy estimates.

Key words convection diffusion reaction equation; pointwise estimate; Green function; energy method

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1 Introduction

The classical one dimensional convection diffusion equation is well known as follows

$$\partial_t c + u\partial_x c = D\partial_x^2 c, \tag{1.1}$$

where $x \in \mathbb{R}$ is the spatial variable, t is the temporal variable, and c represents the material concentration. $D, u > 0$ are constants which are the diffusion coefficient, the current velocity respectively. This equation describes the integrated process of material transport and diffusion. The analytic solution of it can be obtained easily referred to [5], which is a analogue of the heat kernel. So far, many researchers were also interested in the numerical solutions of it, see [2, 13, 16] and the references cited therein. Later, Based on it, Xu and Wang in [22] started to consider the convection diffusion equation with photocatalytic degradation reaction. Then equation (1.1) turns to

$$\partial_t c + u\partial_x c = D\partial_x^2 c + k\partial_{tx}^2 c, \tag{1.2}$$

where $k > 0$ is a constant which is the reaction coefficient. They gave the pointwise estimates of solutions of the Cauchy problem of equation (1.2). The photocatalytic degradation is emerging as a promising technology for degradation of organic contaminants in environmental control.

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It has important applications in water conservancy project, environment engineering, chemical industry, metallurgy, aviation, etc.

In this paper, we are interested in the Cauchy problem of the nonlinear convection diffusion equation with photocatalytic degradation reaction

$$\partial_t c + u\partial_x c = D\partial_x^2 c + k\partial_{tx}^2 c - \partial_x c^2, \tag{1.3}$$

$$c(0,x) = c_0(x), \tag{1.4}$$

we always assume $D > u > 0$ and, without loss of generality, take $k = 1$. It is obvious that a Burger term $\partial_x c^2$ is added to equation (1.2). The problems about solutions become more complicated. Here we mainly are interested in the large time asymptotic behavior of solutions of (1.3)–(1.4) for small initial data.

There were many papers on the large time behavior for the convection diffusion reaction equation and related versions in one dimension (see [3, 4, 9, 10, 14, 15]). The general degenerate convection diffusion reaction equation is

$$c_t + F(c,x,t)_x + H(c,x,t) = G(c,t)_{xx},\ (x,t) \in \mathbb{R} \times (0,T), \tag{1.5}$$

$$c(x,0) = c_0(x), \tag{1.6}$$

where c represents the material concentration and $G(c,t), F(c,x,t), H(c,x,t)$ is the diffusion function, the flux function, the source term respectively. In [14], the authors considered the decay of the weak solutions of (1.5)–(1.6). Specially, in [3, 4, 9, 10, 15], the authors were concerned about the asymptotic form of solutions for large time for varies of specific forms of (1.5)–(1.6). However, It is easy for us to find it impossible to encompass equation (1.3) for equation (1.5) because of the term $\partial_{tx}^2 c$ contained in equation (1.3). Up to our knowledge, there are very few references concerning this kind of equations containing the term $\partial_{tx}^2 c$ like equation (1.3).

The emerging of new term $\partial_{tx}^2 c$ may bring a lot of new researches. In this paper, the term $\partial_{tx}^2 c$ leads to a link between the diffusion coefficient and the current velocity, which compels us only to consider the situation of $D > u$. For the situation of $D \leqslant u$, it remains to be researched. At the same time, it also results in many difficulties in both the analysis of Green function and the energy estimates. In the analysis of Green function, on one hand, it results in the special structure of the eigenvalue, which makes the Green function possess distinct parallel move speed in low frequency and high frequency. On the other hand, it leads the Green function to emerge the singular part in high frequency. Furthermore, in the process to get the global existence of solutions for large initial data, it causes essentially the difficulty because it immediately prevents us to obtain the fundamental energy estimates.

Throughout this paper, we denote by $\mathbb{R}$ the set of real numbers. We also use C to represent the generic constant which may take different values in different places. Let $L^p(\Omega)$ and $W^{m,p}(\Omega)$ be the usual Lebesgue space and Sobolev space endowed with norms $|\cdot|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$ respectively (see e.g. [1]), where

$$|\varphi|_{L^p} := \Big(\int_\Omega |\varphi|^p \mathrm{d}x \Big)^{1/p} \quad \text{and} \quad \|\varphi\|_{W^{m,p}} := \Big(\sum_{|\beta| \leqslant m} \int_\Omega |D^\beta \varphi|^p \mathrm{d}x \Big)^{1/p}.$$

In particular, we take $H^s = W^{s,2}$ with norm $\|\cdot\|_{H^s}$.