Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2017,**37B**(1):97–107
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

LIMITING DIRECTIONS OF JULIA SETS OF ENTIRE SOLUTIONS TO COMPLEX DIFFERENTIAL EQUATIONS*

Jun WANG (王珺)
School of Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail: *majwang@fudan.edu.cn*

Zongxuan CHEN (陈宗煊)†
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
E-mail: *chzx@vip.sina.com*

Abstract In this paper, we mainly investigate the dynamical properties of entire solutions of complex differential equations. With some conditions on coefficients, we prove that the set of common limiting directions of Julia sets of solutions, their derivatives and their primitives must have a definite range of measure.

Key words limiting direction; Julia set; entire function; linear differential equation; deficiency

2010 MR Subject Classification 34M10; 37F10; 30D35

1 Introduction and Main Results

Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a transcendental meromorphic function, where $\mathbb{C}$ is the complex plane and $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. The Fatou set $F(f)$ of $f(z)$ is the subset of $\mathbb{C}$ where the iterates $f^n(z)(n=1,2,\cdots)$ of f are well defined and $\{f^n(z)\}$ forms a normal family. The complement of $F(f)$ is called the Julia set $J(f)$ of $f(z)$. It is well known that $F(f)$ is open, $J(f)$ is closed and non-empty. In general, the Julia set is very complicated. Some basic knowledge of complex dynamics of meromorphic functions can be found in Bergweiler's paper [4] and Zheng's book [29].

The Nevanlinna theory is an important tool in this paper, its usual notations and basic results come mainly from [8, 10, 14]. We use $\lambda(f)$ and $\mu(f)$ to denote the order and the lower order of f respectively, which are defined as [23, Definition 1.6]

$$\lambda(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r},\quad \mu(f)=\liminf_{r\to\infty}\frac{\log^+T(r,f)}{\log r},$$

*Received November 4, 2015. The work was supported by Shanghai Center for Mathematical Sciences, China Scholarship Council (201206105015), National Science Foundation of China (11171119, 11001057, 11571049) and Natural Science Foundation of Guangdong Province in China (2014A030313422).

†Corresponding author: Zongxuan CHEN.

and denote the deficiency of f at the point a by [14, p.46]

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)},\quad \delta(\infty,f)=\liminf_{r\to\infty}\frac{m(r,f)}{T(r,f)}.$$

Baker [2] first observed that for a transcendental entire function $f(z)$, the Julia set can not be contained in any finite set of straight lines, so it can not lie in finitely many rays emanating from the origin. Qiao [18] introduced the definition of limiting direction of $J(f)$, and proved that the Julia set of a transcendental entire function of finite order has infinitely many limiting directions. Here, a limiting direction of $J(f)$ means a limit of the set $\{\arg z_n\,|\,z_n\in J(f)\,\text{is an unbound sequence}\}$. For such limit θ, Zheng say that the Julia set has the radial distribution with respect to the ray $\arg z=\theta$ [27]. Indeed, he need θ to satisfy the requirement that $\Omega(\theta-\varepsilon,\theta+\varepsilon)\cap J(f)$ is unbounded for any $\varepsilon>0$, where

$$\Omega(\theta-\varepsilon,\theta+\varepsilon)=\{z\in\mathbb{C}:\ \arg z\in(\theta-\varepsilon,\theta+\varepsilon)\}.$$

It is easy to see that Qiao's definition and Zheng's definition are the same in essence. Define

$$\Delta(f)=\{\theta\in[0,2\pi):\arg z=\theta\ \text{is a limiting direction of}\ J(f)\}.$$

Clearly, $\Delta(f)$ is closed, by mes$\Delta(f)$ we stand for its linear measure.

If $f(z)$ is a transcendental entire function of finite lower order, Qiao [19] proved that $\text{mes}\,\Delta(f)\geq\min\{2\pi,\pi/\mu(f)\}$. The example in [2] showed that there exists an entire function of infinite lower order whose Julia set has only one limiting direction. Later some observations for a transcendental meromorphic function $f(z)$ were made by [21, 27]: if $\mu(f)<\infty$ and $\delta(\infty,f)>0$, then

$$\text{mes}\,\Delta(f)\geq\min\left\{2\pi,\frac{4}{\mu(f)}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\right\}.$$

Naturally, it is interesting what happens to the limiting directions of meromorphic functions, even entire functions, with infinite lower order. However, since then, there is few answer until Huang and Wang considered entire solutions of the following linear differential equations [12, 13]

$$f^{(n)}+A_{n-1}f^{(n-1)}+\cdots+A_0f=0,\tag{1.1}$$

where $A_i(z)(i=0,1,\cdots,n-1)$ are entire functions of finite lower order. The zeros of transcendental solutions of (1.1) was studied by Lan and Chen [15]. When A_0 is transcendental and $T(r,A_i)=o(T(r,A_0))(i=1,2,\cdots,n-1)$ as $r\to\infty$, every nontrivial solution f of (1.1) is an entire function of infinite lower order, and mes$\Delta(f)\geq\min\{2\pi,\pi/\mu(A_0)\}$ by [13]. By [23, Theorem 1.18], if $\lambda(A_i)<\mu(A_0)(j=1,2,\cdots,n-1)$, then $T(r,A_i)=o(T(r,A_0))(i=1,2,\cdots,n-1)$.

Recently, Zhang et al. [25] investigated the common limiting directions of solutions and their k-th derivative of (1.1), and they obtained $\text{mes}(\Delta(f)\cap\Delta(f^{(k)}))\geq\min\{2\pi,\pi/\mu(A_0)\}$. They also dealt with some second order differential equations, where one coefficient is of small growth, and the other one has a finite deficient value [26]. Indeed, Qiao [20] already pointed out that the Julia set of a transcendental entire function of finite lower order, its derivative and its primitive have a large amount of common radial distribution. His result is stated as follows.

Theorem A Let f be a transcendental entire function of lower order $\mu<\infty$. Then there exists a closed interval $I\subset\mathbb{R}$ such that all $\theta\in I$ are the common limiting directions of $J(f^{(n)})$, $n=0,\pm1,\pm2,\cdots$, and $\text{mes}\,I\geq\min\{2\pi,\pi/\mu\}$. Here $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n\geq0$ or $n<0$, respectively.