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THE LOWER ORDER AND LINEAR ORDER OF MULTIPLE DIRICHLET SERIES*

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Abstract The article investigates the growth of multiple Dirichlet series. The lower order and the linear order of n-tuple Dirichlet series in $\mathbb{C}^n$ are defined and some relations between them and the coefficients and exponents of n-tuple Dirichlet series are obtained.

Key words multiple Dirichlet series; lower order; linear order

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1 Introduction

N-tuple Dirichet series is a function with the following form

$$f(s_1,s_2,\cdots,s_n)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}a_{m_1m_2\cdots m_n}\mathrm{e}^{-\lambda_{m_1}^{(1)}s_1-\lambda_{m_2}^{(2)}s_2\cdots-\lambda_{m_n}^{(n)}s_n}, \tag{1.1}$$

where for every $j\in\{1,2,\cdots,n\}, 0<\{\lambda_{m_j}^{(j)}\}\uparrow+\infty$, $s_j=\sigma_j+\mathrm{i}\tau_j, \sigma_j,\tau_j\in\mathbb{R}(j=1,2,\cdots,n)$.

In order to simplify the form of n-tuple Dirichlet Series, we denote $S=(s_1,s_2,\cdots,s_n)\in\mathbb{C}^n, m=(m_1,m_2,\cdots,m_n)\in\mathbb{N}^n, \lambda_m=(\lambda_{m_1}^{(1)},\lambda_{m_2}^{(2)},\cdots,\lambda_{m_n}^{(n)})\in\mathbb{R}^n$ and $\lambda_mS=(\lambda_{m_1}^{(1)},\lambda_{m_2}^{(2)},\cdots,\lambda_{m_n}^{(n)})\ (s_1,s_2,\cdots,s_n)=\lambda_{m_1}^{(1)}s_1+\lambda_{m_2}^{(2)}s_2+\cdots+\lambda_{m_n}^{(n)}s_n$. The n-tuple Dirichlet series can also be denoted by

$$f(S)=\sum_m a_m\mathrm{e}^{-\lambda_mS}.$$

Let

$$S_m(S)=\sum_m a_j\mathrm{e}^{-\lambda_js}=\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\cdots\sum_{j_n=1}^{m_n}a_{j_1j_2\cdots j_n}\mathrm{e}^{-\lambda_{j_1}^{(1)}s_1-\lambda_{j_2}^{(2)}s_2-\cdots\lambda_{j_n}^{(n)}s_n}.$$

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If $\{S_m(S)\}$ is bounded for a group of complex numbers $(s_1, s_2, \cdots, s_n)$ in $\mathbb{C}^n$ and the limit

$$\lim_{\substack{m_j \to \infty \\ j=1,2,\cdots,n}} S_m(S)$$

exists, then we say that series (1.1) is boundedly convergent for the group of numbers, and the limit is called as the sum of series (1.1). If $\{S_m(\sigma_{1_0} + \mathrm{i}\tau_1, \sigma_{2_0} + \mathrm{i}\tau_2, \cdots, \sigma_{n_0} + \mathrm{i}\tau_n)\}(m \in \mathbb{N}^n)$ converges uniformly with respect to $-\infty < \tau_j < \infty$ $(j = 1, 2, \cdots, n)$, and the limit

$$\lim_{m\to\infty^n} S_m(\sigma_{1_0} + \mathrm{i}\tau_1, \sigma_{2_0} + \mathrm{i}\tau_2, \cdots, \sigma_{n_0} + \mathrm{i}\tau_n)$$

exists uniformly with respect to $-\infty < \tau_j < \infty$ $(j = 1, 2, \cdots, n)$, then we say that series (1.1) is uniformly and boundedly convergent on $(\sigma_1 = \sigma_{1_0}, \sigma_2 = \sigma_{2_0}, \cdots, \sigma_n = \sigma_{n_0})$. And if the limit

$$\lim_{m\to\infty^n} \sum_m |a_j \mathrm{e}^{-\lambda_j S}|$$

exists for a group of complex numbers $(s_1, s_2, \cdots, s_n)$, then we say that series (1.1) is absolutely convergent for the group of numbers. Series (1.1) has grouped relative boundedly convergent abscissas $(\sigma_{1_b}, \sigma_{2_b}, \cdots, \sigma_{n_b})$, grouped relative uniformly and boundedly convergent abscissas $(\sigma_{1_u}, \sigma_{2_u}, \cdots, \sigma_{n_u})$, and grouped relative absolutely convergent abscissas $(\sigma_{1_a}, \sigma_{2_a}, \cdots, \sigma_{n_a})$, where these numbers of $\sigma_{1_b}, \sigma_{2_b}, \cdots, \sigma_{n_b}, \sigma_{1_u}, \sigma_{2_u}, \cdots, \sigma_{n_u}, \sigma_{1_a}, \sigma_{2_a}, \cdots, \sigma_{n_a}$ can be either finite or infinite. When $\sigma_1 > \sigma_{1_b}, \sigma_2 > \sigma_{2_b}, \cdots, \sigma_n > \sigma_{n_b}$, series (1.1) is boundedly convergent; while $\sigma_1 < \sigma_{1_b}, \sigma_2 < \sigma_{2_b}, \cdots, \sigma_n < \sigma_{n_b}$, series (1.1) is not boundedly convergent. When $\sigma_1 = \sigma_{1_0} > \sigma_{1_u}, \sigma_2 = \sigma_{2_0} > \sigma_{2_u}, \cdots, \sigma_n = \sigma_{n_0} > \sigma_{n_u}$, series (1.1) is uniformly and boundedly convergent on $\sigma_1 = \sigma_{1_0}, \sigma_2 = \sigma_{2_0}, \cdots, \sigma_n = \sigma_{n_0}$; while $\sigma_1 = \sigma_{1_1} < \sigma_{1_u}, \sigma_2 = \sigma_{2_1} < \sigma_{2_u}, \cdots, \sigma_n = \sigma_{n_1} < \sigma_{n_u}$, series (1.1) is not uniformly and boundedly convergent on $\sigma_1 = \sigma_{1_1}, \sigma_2 = \sigma_{2_1}, \cdots, \sigma_n = \sigma_{n_1}$. When $\sigma_1 > \sigma_{1_a}, \sigma_2 > \sigma_{2_a}, \cdots, \sigma_n > \sigma_{n_a}$, series (1.1) is absolutely convergent, while $\sigma_1 < \sigma_{1_b}, \sigma_2 < \sigma_{2_b}, \cdots, \sigma_n < \sigma_{n_b}$, series (1.1) is not absolutely convergent.

We build an n-dimension space of $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ and denote the relative boundedly convergent abscissas, relative uniformly and boundedly convergent abscissas, and relative absolutely convergent abscissas by the points of the space. Now we make a straight line in the space as follows:

$$\begin{cases} \sigma_1 = r\Phi_1(\phi) = r \sin\phi_{n-1} \sin\phi_{n-2} \cdots \sin\phi_2 \sin\phi_1 + c_1; \\ \sigma_2 = r\Phi_2(\phi) = r \sin\phi_{n-1} \sin\phi_{n-2} \cdots \sin\phi_2 \cos\phi_1 + c_2; \\ \qquad \cdots \\ \sigma_{n-1} = r\Phi_{n-1}(\phi) = r \sin\phi_{n-1} \cos\phi_{n-2} + c_{n-1}; \\ \sigma_n = r\Phi_n(\phi) = r \cos\phi_{n-1}, \end{cases}$$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_{n-1}) \in (0, \frac{\pi}{2})^{n-1}, C = (c_1, c_2, \cdots, c_{n-1}, 0) \in \mathbb{C}^n$ and $r \in \mathbb{R}$ is a varied parameter.

We denote the relative boundedly convergent abscissas on the line by

$$\sigma_b(C, \phi) = (\sigma_{1_b}(C, \phi), \sigma_{2_b}(C, \phi), \cdots, \sigma_{n_b}(C, \phi)),$$

the relative uniformly and boundedly convergent abscissas by

$$\sigma_u(C, \phi) = (\sigma_{1_u}(C, \phi), \sigma_{2_u}(C, \phi), \cdots, \sigma_{n_u}(C, \phi)),$$