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SOLVABILITY OF AN IMPLICIT FRACTIONAL INTEGRAL EQUATION VIA A MEASURE OF NONCOMPACTNESS ARGUMENT*

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Abstract In this paper, we study the existence of solutions to an implicit functional equation involving a fractional integral with respect to a certain function, which generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral. We establish an existence result to such kind of equations using a generalized version of Darbo's theorem associated to a certain measure of noncompactness. Some examples are presented.

Key words umplicit fractional integral equation; measure of noncompactness; Darbo's theorem

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1 Introduction

The measure of noncompactness argument is an important tool in Nonlinear Functional Analysis. Many authors used such argument to study the existence of solutions to various classes of nonlinear integral equations. We invite the reader to consult the papers [1, 3–13] and the references therein.

Let $\mathbb{V}$ be a Banach space with respect to a certain norm $\|\cdot\|$. Let $\mathbb{B}$ be the set of all nonempty bounded subsets of $\mathbb{V}$. We say that $\sigma : \mathbb{B} \to [0, \infty)$ is a measure of noncompactness (see [4]) if the following conditions hold:

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(C1) for every $B \in \mathbb{B}$,

$$\sigma(B) = 0 \Longrightarrow B \text{ is precompact;}$$

(C2) for every pair $(Q_1, Q_2) \in \mathbb{B} \times \mathbb{B}$, we have

$$Q_1 \subseteq Q_2 \Longrightarrow \sigma(Q_1) \leq \sigma(Q_1);$$

(C3) for every $B \in \mathbb{B}$,

$$\sigma(\overline{B}) = \sigma(B) = \sigma(\overline{\mathrm{co}}B),$$

where $\overline{\mathrm{co}}B$ denotes the closed convex hull of B;

(C4) for every pair $(Q_1, Q_2) \in \mathbb{B} \times \mathbb{B}$ and $\gamma \in (0,1)$, we have

$$\sigma(\gamma Q_1 + (1-\gamma)Q_2) \leq \gamma\sigma(Q_1) + (1-\gamma)\sigma(Q_2);$$

(C5) if $\{Q_n\}$ is a sequence of closed and decreasing (w.r.t $\subseteq$) sets in $\mathbb{B}$ such that $\sigma(Q_n) \to 0$ as $n \to \infty$, then $Q_\infty := \bigcap\limits_{n=1}^{\infty} Q_n$ is nonempty and compact.

Let Φ be the set of functions $\varphi : [0,\infty) \to [0,\infty)$ such that

(Φ_1) φ is a nondecreasing function;

(Φ_2) φ is an upper semi-continuous function;

(Φ_3) $\varphi(s) < s$, for all $s > 0$.

Our main tool in this paper is the following fixed point result established in [3], which is a generalization of the well-known Darbo's theorem [4].

Lemma 1.1 Let $\mathcal{Z}$ be a nonempty, bounded, closed and convex subset of the Banach space $\mathbb{V}$. Let $D : \mathcal{Z} \to \mathcal{Z}$ be a continuous mapping such that

$$\sigma(DW) \leq \varphi(\sigma(W)), \quad W \in \mathcal{P}(\mathcal{Z}),$$

where $\varphi \in \Phi$ and $\mathcal{P}(\mathcal{Z})$ denotes the set of all the subsets of $\mathcal{Z}$. Then D has at least one fixed point.

Observe that Darbo's theorem follows from Lemma 1.1 by taking $\varphi(t) = kt$, $0 \leq k < 1$.

In this paper, we are concerned with the existence of solutions to the following implicit integral equation

$$y(t) = F\left(t, y(t), \psi\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t,s,y(s))\mathrm{d}s\right)\right), \quad t \in [a,T], \tag{1.1}$$

where $T > 0$, $a \geq 0$, $\alpha \in (0,1)$, $F : [a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$, $g : [a,T] \to \mathbb{R}$ and $h : [a,T] \times [a,T] \times \mathbb{R} \to \mathbb{R}$. Observe that we can write (1.1) in the form

$$y(t) = F\left(t, y(t), \psi\left(\Gamma(\alpha) I_{a,g}^{\alpha} h(t,\cdot,y(\cdot))(t)\right)\right), \quad t \in [a,T],$$

where $I_{a,g}^{\alpha}$ is the fractional integral of order α with respect to the function g defined by (see [14])

$$I_{a,g}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{(g(t)-g(s))^{1-\alpha}}\mathrm{d}s, \quad t \in [a,T].$$

Note that for $g(s) = s$, $I_{a,g}^{\alpha}$ is the Riemann-Liouville fractional integral of order α defined by (see [14])

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}\mathrm{d}s, \quad t \in [a,T].$$