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WAVELET-BASED ESTIMATOR FOR THE HURST PARAMETERS OF FRACTIONAL BROWNIAN SHEET *

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Abstract It is proposed a class of statistical estimators $\hat{H} = (\hat{H}_1, \cdots, \hat{H}_d)$ for the Hurst parameters $H = (H_1, \cdots, H_d)$ of fractional Brownian field via multi-dimensional wavelet analysis and least squares, which are asymptotically normal. These estimators can be used to detect self-similarity and long-range dependence in multi-dimensional signals, which is important in texture classification and improvement of diffusion tensor imaging (DTI) of nuclear magnetic resonance (NMR). Some fractional Brownian sheets will be simulated and the simulated data are used to validate these estimators. We find that when $H_i \geq 1/2$, the estimators are accurate, and when $H_i < 1/2$, there are some bias.

Key words detection of long-range dependence; self-similarity; Hurst parameters; wavelet analysis; fractional Brownian sheet

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1 Introduction

In numerous fields such as hydrology, biology, telecommunication and economics, the data available for analysis usually have scaling behavior (or long-range dependence and self-similarity) that need to be detected. The key point for detecting scaling behavior is the estimation of the Hurst parameters H, which are used in characterizing self-similarity and long-range dependence. The literature on the topic is vast (see e.g. [1–3]). Most of the published research concerns the scaling behavior in 1D data. However, it is also important to detect the scaling behavior in multi-dimensional signals, which is important in texture classification and

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improvement of diffusion tensor imaging (DTI) of nuclear magnetic resonance (NMR). Moreover, many multidimensional data from various scientific areas also have anisotropic nature in the sense that they have different geometric and probabilistic along different directions. The study of long-range dependence and self-similarity in multidimensional case is usually based on the model of anisotropy fractional Gaussian fields such as fractional Brownian sheet (fBs) [4, 5], operator scaling Gaussian random field (OSGRF) [6] and extended fractional Brownian field (EFBF) [7].

We focus on fBs, which plays an important role in modeling anisotropy random fields with self-similarity and long-range dependence (see e.g. [5, 8, 9]) since it was first introduced by Kamont [4]. Fractional Brownian sheet has arisen in the study of texture analysis for classification [10, 11]. Besides research on the texture analysis, fractional Brownian sheets can also be used to drive stochastic partial differential equations [12]. Moreover, Diffusion-tensor images (DTI) of nuclear magnetic resonance (NMR) in brain is now based on three-dimensional Brownian motion [13] $H_1 = H_2 = H_3 = \frac{1}{2}$. Compared with this model, fractional Brownian sheet may be a better model because it can describe the long-range dependence, which may exist in brain.

The main purpose of this paper is to estimate the Hurst parameters of fractional Brownian sheets via wavelet analysis. Wavelet analysis is an efficient tool for the estimation of Hurst parameter, but most of the published research on the topic focus on the 1D case. Wavelet-based estimator for the Hurst parameter was first introduced by Flandrin [1]. Abry et al. studied this estimator in the case of single-parameter scaling processes (self-similarity and long-range dependence) [2, 3]. Coeurjolly et al. studied the multivariate fractional Brownian motion using wavelet analysis [14, 15]. Such estimators are based on the wavelet expansion of the scaling process and a regression on the log-variance of wavelet coefficient. It was usually assumed that wavelet coefficients are uncorrelated within and across octaves. Such wavelet-based estimator for the Hurst parameters of fBs can be seen in [16]. Bardet [17] and Morales [18] removed this assumption and proved the asymptotic normality of these estimators in the case of fractional Brownian motion (fBm), proposed a two-step estimator, which has the lowest variance in all of the least square estimators. Compared with other estimators, such as the R/S method, the periodogram, the maximum-likelihood estimation, and so on, the wavelet-based estimator performs better in both the statistical and computational senses, and is superior in robustness (see [2, 3] and the references therein). It not only has small bias and low variance but also leads to a simple, low-cost, scalable algorithm. Besides, the wavelet-based method can also eliminate some trends (linear trends, polynomial trend, or more) by the property of its vanishing moments [2], which makes the estimator robust to some nonstationarities.

The paper is organized as follows. In Section 2, some properties of Gaussian random variables and some limiting theorems used in this paper are introduced. In Section 3, we obtain some properties of wavelet coefficient of fBs (see Proposition 3.1). Based on these properties, A class of estimators $\hat{H}$ for the Hurst parameter of fBs are proposed (see Section 3.2). We also prove that these estimators $\hat{H}$ are asymptotically normal following the approach in [18] (see Theorem 3.5 and Section 3.3). Using the two-step procedure that proposed by Bardet [19], we realize the generalized least squares (GLS) estimator, which has the lowest variance of $\hat{H}$ (see Section 3.4). Finally, this two-step estimator $\hat{H}_{og}$ is applied to the synthetic fBs generated by